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Abstract

An economy with clean and dirty intermediate inputs may fall into a trap characterized by low environmental quality and low life expectancy, while the others converge to opposite steady states. We propose new strategies towards sustainable growth. They include: (i) taxes (subsidies) imposed on the production of intermediate inputs to improve environmental quality, and therefore, life expectancy and capital accumulation, in order to guarantee that an economy locked in a poverty-environment trap can escape the stagnation; (ii) taxes (subsidies) imposed on the production of intermediate inputs, consumption, and capital income in order to decentralize the transition to the social optimum.

JEL-Code: D620, E220, H210, H230, K320.

Keywords: overlapping generations economy, poverty-environment trap, intermediate sectors, final good sectors, fiscal policy.

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1 Introduction

We consider an economy in which environmental quality is degraded by the production of dirty intermediate inputs. If the people in such an economy expect to live longer, they have incentive to save more for consumption when old. On one hand, higher savings foster economic growth through the capital accumulation channel, but on the other hand, degrade the environment through producing dirty intermediate inputs that negatively affect the longevity of the next generation, thus discouraging them to save. Therefore, as will be shown in our paper, depending on the initial conditions and the government's policies for allocating capital towards clean and dirty intermediate inputs, an economy may converge to a low environmental quality steady state associated with short life expectancy or to a high environmental quality one with high life expectancy. That is to say, the government's policies play a crucial role in attaining both environmental protection and good economic performance. A tax imposed on producing dirty intermediate inputs with a subsidy for producing clean inputs is necessary for an economy to escape a poverty-environment trap. Besides the tax (and subsidy) on intermediate inputs, other taxes on consumption and capital income must be introduced in order to improve social welfare. This paper explores these policies.

Indeed, environmental issues have received widespread attention from scientists, politicians, and media for over last two decades. The call for a sustainable development strategy has emerged in the context of climate change. There is a common agreement that the economic process generates negative environmental externalities on economic well-being. The requirement for a sustainable development strategy itself needs a solid theoretical background which links long-run economic growth with environmental quality. A micro-foundation-macroeconomic model that captures the crucial issues above should be constructed. Therefore, extending Diamond's (1965) overlapping generations model to include environmental factors seems suitable for this research purpose. John and Pecchennio (1994) produced the first theoretical paper linking environmental aspects, which are included in the household's utility function, with long-run economic growth in the overlapping generations framework. The authors point out the potential conflict between economic growth and environmental quality and explain the positive correlation between environmental quality and growth, implying that there is a divergence across countries in terms of environmental quality and income. The paper of John and Pecchennio (1994) paved the way for many subsequent papers that call for alternative Pareto-improvement policies in order to decentralize the social optimum. An incomplete list of these papers include John et al. (1995), Ono (1996; 2003), Jouvet et al. (2000), Gutierrez (2008), and recently Dao and Dávila (2014). These papers, however, ignore the role of life expectancy, which is shown in the paper at hand and other papers in the literature to play a crucial role in capital accumulation, hence, it may be a possible cause of the poverty trap.

Chakraborty (2004) introduces endogenous life expectancy in an overlapping generations model to show that shorter life expectancy discourages savings, thus showng that high mortality societies may be locked in a development trap. Many papers have adapted Chakraborty's (2004) ideas by assuming life expectancy to be endogenously determined by environmental quality or pollution stock in order to study the interactions between environment, life expectancy, and growth (see, for example, Jouvet et al. 2010, Mariani et al. 2010, Palivos and Varvarigos 2010, Goenka et al. 2012, Raffin and Seegmuller 2012, Fodha and Seegmuller 2014, Varvarigos 2011; 2014). The arguments for the dependence of life expectancy on the environment are confirmed by many empirical studies in medicine and epidemiology (see Pope 2000, Pope et al. 2004, Evan and Smith 2005, and Pimentel et al. 2007). Stylized fact 1 in the next section also shows a strong positive correlation between the environmental quality score and life expectancy across 135 countries, thus supporting the link between the two variables.

The paper at hand is in line with previous theoretical literature that has adapted the idea that life expectancy is endogenously determined by environmental quality in order to explain the recurrent stylized fact of convergence clubs, as presented in the next section, and propose fiscal strategies for escaping poverty-environment traps towards sustainable growth in the long run. Our paper, however, departs from the related literature in at least two important aspects. First, unlike previous papers that assume the environment regenerates or degrades itself with a constant rate, we employ the laws of thermodynamics —first introduced to environmental economics by Georgescu-Rogen (1971)— to set up the general form of environmental dynamics. Indeed, this approach unifies most of the specific dynamics of environment in the literature.[2] In this way, we show the possibilities for the existence of more than one poverty trap and even continuums of steady states.[3] This is one of the crucial results of our paper. Second, although many papers in the related literature have also pointed out the possibility for the existence of one poverty trap, the policies (of reallocating resources for production between sectors) which are designed to help an economy escape (or avoid) such a poverty trap and converge to the social optimum in a framework of general equilibrium are still lacking. Our paper attempts to fill this crucial gap in the literature.

In addition to the two crucial aspects pointed out above, we also review and clarify our departures from the closest literature. Indeed, Varvarigos (2010) mentions the distribution of public spending between pollution abatement and public health which improves the longevity of the agents, and thus, encourages their savings, in order to maximize the interior equilibrium capital ratio in the long run. When a poverty trap exists, this distribution strategy minimizes the thresholds of the capital ratio, below which the economy is led towards a poverty trap. In a similar framework with endogenous technological progress driven by the externality of capital accumulation, Pavilos and Varvarigos (2010) highlight the role of an active pollution abatement policy as an engine for long-run growth and help the economy avoid a poverty trap. Note that, the policies in Varvarigos (2010) and Pavilos and Vavarigos (2010) may help an economy to *avoid* the poverty trap but *cannot* help the economy to *escape* if it is already locked in (or even very closed to) the trap.[4] Although most papers in the closest literature (e.g. Varvarigos 2010, Pavilos and Vavarigos 2010, Goenka et al. 2012, and Fodha and and Seegmuller 2014) study the policies to improve the social welfare, a policy that leads to the social optimum is lacking in these papers. The paper at hand tries to fills these two gaps in one theoretical model by studying not only the strategy to escape/avoid a poverty-environment trap but also the strategy that leads to the social optimum.

In this paper, we set up a two-period overlapping generations model in which the agents get utility from their consumption (when young and old) and their life expectancy, which is endogenously determined by the environmental quality in the first period of their lives. Following Aghion and Howitt (2009) and Acemoglu et al. (2012), we also introduce intermediate sectors in our model to find room for reallocating capital between these sectors towards Pareto improvement. There are two intermediate input sectors, clean and dirty, and one final consumption good. We show that in a competitive economy, the allocation of capital for each intermediate sector depends linearly on its relative productivity versus the aggregate productivity of the economy in producing the final output. We assume that only producing the dirty intermediate input degrades the environment. The dynamic interactions between environment and capital accumulation/allocation are

[2]Bovenberg and Smulders (1995) and Smulders (1995a,b) follow this approach to the dynamics of the environment in variations of the growth model in order to study environmental policies for sustainable long-run growth. However, these papers ignore the impact of the environment on life expectancy, as well as the crucial role of life expectancy on capital accumulation.

[3]Most papers in the related literature that link life expectancy with the environment, by assuming that the environment regenerates or degrades itself with a constant rate, show the possibility for the existence of one poverty trap (see Mariani et al. 2010, Goenka et al. 2012, Raffin and Seegmuller 2012, Fodha and Seegmuller 2014, Varvarigos 2011, 2014).

[4]Mariani et al. (2010) also discuss some ways to escape the poverty trap. However in general, their ways exogenously increase the response of the life expectancy function to the environmental quality. In a different optimal growth model with exhaustible resources and convex-concave production technology, Le Van et al. (2010) study strategies for extracting natural resources in order to help a developing country escape its poverty trap. The strategies depend on interactions between the technology and the impatience of the economy, the characteristics of the resource revenue function, the abundance of resources, and the initial level of capital per capita. This paper, however, completely ignores the environmental aspects of extracting natural resources and their impacts on economic well-being which may capture the impatience of the agents.

central to our analysis. Depending on the characteristics of the regeneration of the environment and the surviving probability functions, our model exhibits multiple equilibrium steady states and even possibilities for continuums of steady states which may explain the existence of poverty-environment traps and the segmentation of development across countries as a stylized fact which we will introduce in the next section. This is one of the crucial results of our paper. After studying the dynamics and convergences to the steady states, we propose balanced fiscal strategies towards sustainable growth in the long run. To the best of our knowledge, these strategies are quite new compared to the existing literature. In particular, for an economy locked in a poverty-environment trap, we propose a set of taxes and subsidies on the production of dirty and clean intermediate inputs, respectively, in order to free the economy from stagnation. When the economy has escaped the poverty-environment trap, we introduce a set of taxes (and/or subsidies) on the production of intermediate inputs, consumption, and capital income to implement the social optimum during the transition as a competitive outcome. Note that this set of taxes (and/or subsidies) is also applied to the economies that are not locked in poverty-environment traps to decentralize their social optimum.

The rest of the paper is organized as follows. The next section provides some stylized facts on environmental quality and life expectancy, as well as the segmentation of development across countries. The benchmark version of our model is presented in Section 3. Section 4 studies the competitive equilibria and dynamics of the economies. Section 5 studies the steady states and their stability properties. We address the problem of the benevolent social planner and characterize the first-best steady state in Section 6. The strategy for an economy to escape a poverty-environment trap is presented in Section 7 and the decentralization of the social optimum is discussed in Section 8. Some extensions for the cases of there being more than 3 distinct steady states and/or continuum of steady state are discussed in Section 9. Finally, Section 10 concludes the paper.

2 Stylized facts

Mariani et al. (2010) provide a stylized fact across 132 countries showing a strong positive relationship between the Environmental Performance Index (EPI) score and life expectancy. The authors also provide the second stylized fact that EPI and life expectancy are bimodally distributed across these countries, suggesting the possibility for a poverty-environment trap characterized by low environmental quality, and hence, short life expectancy. In this section, we reintroduce and *enrich* these stylized facts with more updated data and a larger size of 135 countries in which the EPI score is constructed comprehensively from 20 indicators reflecting national-level environmental data. EPI covers two objectives: (i) environmental health as determined by child mortality, air quality, drinking water and sanitation; and (ii) ecosystem vitality as determined by water resources, biodiversity and habitat, forest, and many others.[5] In the second stylized fact, Mariani et al. (2010) point out two clubs of convergence in terms of both environmental quality and life expectancy. In this section, we show that there may be *even* more than such two clubs of convergence which may reflect the more complex segmentations of development across countries. This stylized fact will be matched by our theoretical results.

Stylized fact 1: *Strong positive correlation between life expectancy and environmental quality across countries.*

Figure 1 provides a strong positive correlation between environmental quality and life expectancy, thus supporting the idea that environment has a positive impact on longevity.

[5]For more information and methodology about calculating EPI score, see Environmental Performance Index at http://epi.yale.edu/our-methods.

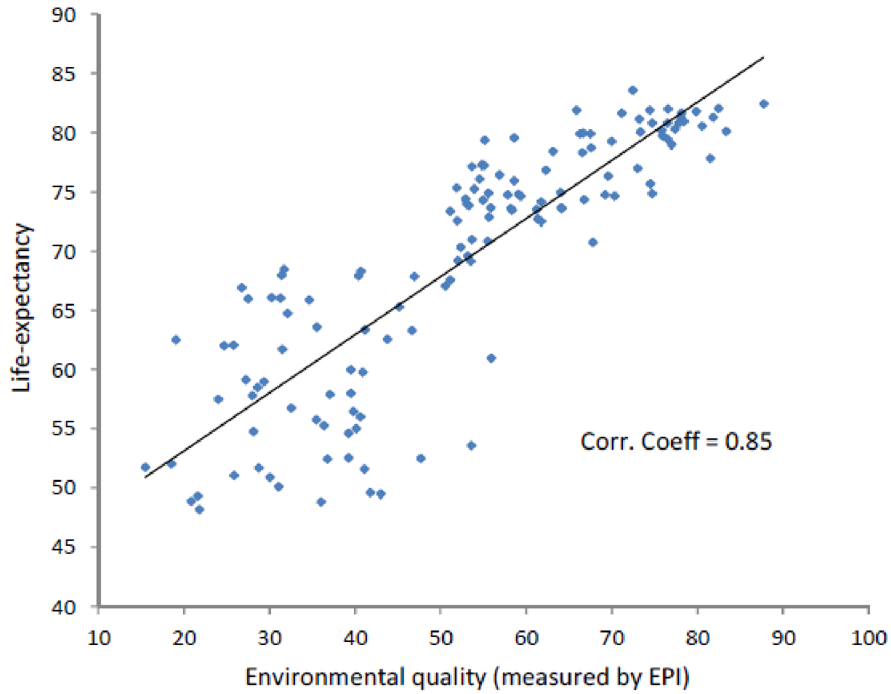

Fig 1. Environmental quality and life expectancy

Sources: Yale Center for Environmental Law and Policy (2014), United Nations (2013)

Stylized fact 2: *Clubs of convergence across countries.*

Figures 2a and 2b exhibit histograms and kernel density estimates with optimal bandwidths of both EPI and life expectancy with the choices of kernel functions Epanechikov and Biweight, respectively. These figures may depict multimodal distributions of both environmental quality and life expectancy, suggesting the possibility for poverty-environment traps.

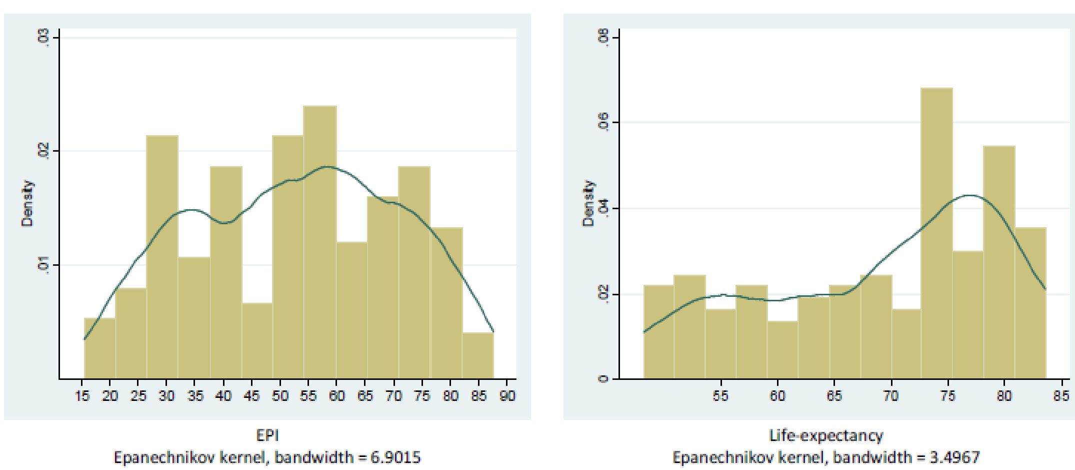

Fig 2a. Histograms and (Epanechnikov kernel estimated) distributions of EPI and life expectancy

Sources: Yale Center for Environmental Law and Policy (2014), United Nations (2013)

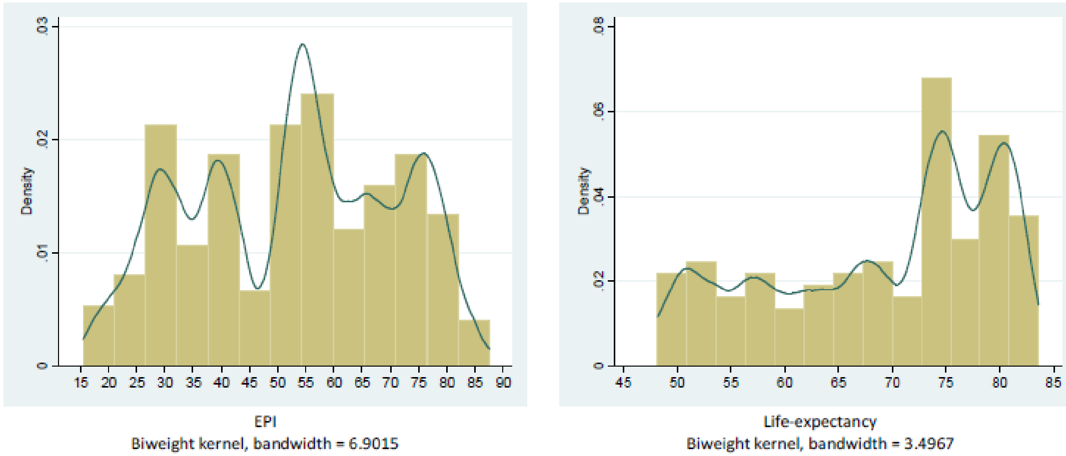

Fig 2b. Histograms and (Biweight kernel estimated) distributions of EPI and life expectancy

Sources: Yale Center for Environmental Law and Policy (2014), United Nations (2013)

Indeed, for the entire sample, we use the Hartigans' *dip* test[6] to reject the null hypotheses of unimodality for distributions of both EPI and life expectancy. In effect, for EPI data, we compute the dip test statistic $d_{EPI} = 0.0375$. Hence, with our sample size of 135 observations, we can infer from Hartigan and Hartigan (1985) that the null hypothesis of unimodality for the distribution of EPI is rejected (at the 10% level of significance) since 0.0375 is greater than the refereed critical value 0.0341. Similarly, that of life expectancy is rejected (at around 5% level of significance) since the dip test statistic of life expectancy $d = 0.0406$ is greater than the (corresponding) critical value 0.0370. That is to say double multimodality of EPI and life expectancy in distributions can be explained as a poverty-environment trap. In effect, we divide both distributions into two groups with equal sizes. Since the number of observations is 135, which is odd, then the 68th observation of each distribution is the median. So, the median of EPI distribution is $m_{EPI} = 53.61$ and that of life expectancy is $m_{LE} = 72.9$. We find that: (i) 62 out of 67 countries with EPI $< m_{EPI}$ also fall within the group characterized by life expectancy below m_{LE}; and (ii) 63 out of 67 countries with EPI $> m_{EPI}$ belong to the group characterized by life expectancy above m_{LE}.

Now it is more interesting to divide the distribution of life expectancy into three groups with equal sizes of 45 observations. Group 1 consists of countries with the lowest life expectancy and group 3 consists of thoses with the highest life expectancy. We construct subsamples that combine two out of these three groups and we proceed with the *dip* tests to measure the multimodality of each subsample. The statistical results in Table 1 suggest that the null hypotheses of the unimodality of each subsample are rejected significantly.

Sample	dip test statistics	
	EPI	Life-expectancy
Groups 1 and 2	.0572**	.0956***
Groups 2 and 3	.0359	.0608***

*,**, and *** imply the null hypothesis of unimodality is rejected at 10%, 5%, and 1% levels of significance, respectively.

Table 1. "dip" test for multimodality

[6] "The dip test measures multimodality in a sample by the maximum difference, over all sample points, between the empirical distribution function and the unimodal distribution function that minimizes that maximum difference" (Hartigan and Hartigan 1985, p70).

The results in Table 1 suggest that there may be more than two clubs of convergence in both environment and life expectancy that correspond to the segmentations of development across countries. This hypothesis seems well matched with the Biweight kernel estimated distributions as depicted in the Figure 2b. Interestingly, the theoretical results of our paper also point out the possibilities for the similar occurrence.

3 The benchmark model

We consider a discrete time overlapping generations economy with a constant population of identical agents and environmental quality affecting the longevity (life expectancy) of the old agents. In any period $t \in \mathbb{N}$, the final output is produced out of intermediate inputs and labor under a perfectly competitive environment. We assume that there are two kinds of intermediate inputs, clean and dirty, that are denoted by "*c*" and "*d*", respectively. Only the producing dirty intermediate inputs pollutes the environment. The monopolist of the intermediate sector $i \in \{c, d\}$ chooses the amount of intermediate good i to be produced so as to maximize its monopoly profit.

3.1 Final good sector

The final good is produced under perfect competition according to the following production function

$$Y_t = L^{1-\alpha}\left(A_c^{1-\alpha}x_{ct}^{\alpha} + A_d^{1-\alpha}x_{dt}^{\alpha}\right); \qquad \alpha \in (0,1) \tag{1}$$

where Y_t is the total final output; x_{ct}^{α} and x_{dt}^{α} are the amounts of clean input and dirty input, respectively, in period t, and A_c, A_d reflect their corresponding total factor productivities; L is the aggregate labor which is the population size of the young generation of the economy.[7]

Without loss of generality, we normalize the population size of each young generation $L = 1$. Therefore, the return to labor and price of intermediate inputs i are, respectively:

$$w_t = (1-\alpha)\left(A_c^{1-\alpha}x_{ct}^{\alpha} + A_d^{1-\alpha}x_{dt}^{\alpha}\right) \tag{2}$$

$$p_{it} = \alpha A_i^{1-\alpha}x_{it}^{\alpha-1}; \qquad i \in \{c, d\} \tag{3}$$

3.2 Intermediate input sectors

For the sake of simplicity, we assume that each intermediate input $i \in \{c, d\}$ is produced according to the following production function

$$x_{it} = k_{it} \tag{4}$$

where k_{it} is the amount of physical capital used as the input in the intermediate sector i.

We assume that the physical capital fully depreciates in each period. So, the cost of producing x_{it} units of intermediate good i is $r_t k_{it}$, where r_t is the rental rate of capital in period t. The monopolist of the intermediate good i decides the quantity x_{it} to be produced so as to maximize her monopoly profit, therefore the monopoly profit of sector i is

$$\pi_{it} = \max_{x_{it}} p_{it}x_{it} - r_t x_{it} \tag{5}$$

Substituting (3) and (4) into (5) we have

[7] We can extend the production function without changing the qualitative analyses crucially using a more general form $Y_t = L^{1-\alpha}\int_0^1 a_j^{1-\alpha}x_{jt}^{\alpha}dj$.

$$\pi_{it} = \max_{k_{it}} \alpha A_i^{1-\alpha} k_{it}^{\alpha} - r_t k_{it} \tag{6}$$

Hence,

$$k_{it} = \left(\frac{\alpha^2}{r_t}\right)^{\frac{1}{1-\alpha}} A_i \tag{7}$$

The aggregate physical capital in period t is

$$k_t = k_{ct} + k_{dt} = \left(\frac{\alpha^2}{r_t}\right)^{\frac{1}{1-\alpha}} (A_c + A_d) = \left(\frac{\alpha^2}{r_t}\right)^{\frac{1}{1-\alpha}} A \tag{8}$$

where $A = A_c + A_d$ is the aggregate total factor productivity (or the technological level of the economy).

So, the rental rate of capital is

$$r_t = \alpha^2 \left(\frac{k_t}{A}\right)^{\alpha-1} \tag{9}$$

and the amount of capital allocated to produce the intermediate input $i \in \{c, d\}$ is

$$k_{it} = \frac{A_i}{A} k_t \tag{10}$$

and the amount of final output in period t is

$$Y_t = A_c^{1-\alpha} x_{ct}^{\alpha} + A_d^{1-\alpha} x_{dt}^{\alpha} = A_c^{1-\alpha} k_{ct}^{\alpha} + A_d^{1-\alpha} k_{dt}^{\alpha} = \frac{A_c}{A^{\alpha}} k_t^{\alpha} + \frac{A_d}{A^{\alpha}} k_t^{\alpha} = A^{1-\alpha} k_t^{\alpha} \tag{11}$$

It is straightforward to find that the allocation rule of capital in (10) maximizes the quantity of the final output because this allocation rule equalizes the marginal productivities of the intermediate inputs.[8]

The monopoly profit of the intermediate sector i is

$$\pi_{it} = \alpha(1-\alpha) A_i \left(\frac{k_t}{A}\right)^{\alpha} \tag{12}$$

We assume that the intermediate producing firms are owned by the contemporary generation of young agents. Therefore, the monopoly profits are distributed to all contemporary young agents. Hence, the total income of the representative young agent in period t is

$$I_t = \pi_{ct} + \pi_{dt} + w_t = (1-\alpha^2) A^{1-\alpha} k_t^{\alpha} \tag{13}$$

3.3 Pollution and environmental quality

We assume that only the dirty intermediate inputs pollute the environment, and environmental quality evolves according to

$$E_t = E_{t-1} + \psi(E_{t-1}) - \xi k_{dt} \tag{14}$$

[8] In effect, given a stock of capital k_t, the problem of maximizing the final output is equivalent to $\max_{k_{ct}, k_{dt}} A_c^{1-\alpha} k_{ct}^{\alpha} + A_d^{1-\alpha} k_{dt}^{\alpha}$ s.t. $k_{ct} + k_{dt} = k_t$, which gives us exactly the rule of capital allocation in (10).

where $\xi > 0$ is the rate of environmental degradation of dirty intermediate input and ξk_{dt} is the aggregate pollution resulting from the production of k_{dt} units of intermediate input d; $E_t \in \mathbb{R}$ is the environmental quality index in period t; and $\psi(E)$ is the regeneration of the environment depending on the quality of the environment itself.

The regeneration of the environment $\psi(E)$ takes different forms in the literature.[9] In the paper at hand, we follow the laws of thermodynamics as mentioned in Georgescu-Rogen (1971)—the first paper introduces the laws of thermodynamics to economics— in order to set up the general form of $\psi(E)$. This is a crucial feature of our paper. Indeed, the state of the environment is constrained by biophysical principles, specifically through two processes: (i) the *entropic process* and (ii) the *preservation process*. The Earth or an economy is basically a *closed* system with respect to material. According to the *law of material or energy conservation*, material is neither lost nor created in any transformation process. The *entropic process* transforms the availability of material or energy in a closed system in the sense that the available energy is continuously transformed into unavailable energy until it disappears completely. *"All kinds of energy are gradually transformed into heat and heat becomes so dissipated in the end that man can no longer use it"* (Georgescu-Roegen 1975, p.352). We are fortunate that our Earth is an *open* system with respect to energy. The *preservation process* refers to the constant receiving of solar radiation, which provides energy to compensate for the entropic process on the Earth, thus making resources renewable. That is to say, while natural and human transformation processes destroy the availability of material, new energy inflows provide energy to recollect material and energy and to offset the destruction. This explains the equilibrium in our ecology systems and the renewable nature of natural resources (Smulders 1995). These natural processes are depicted in the figures below.

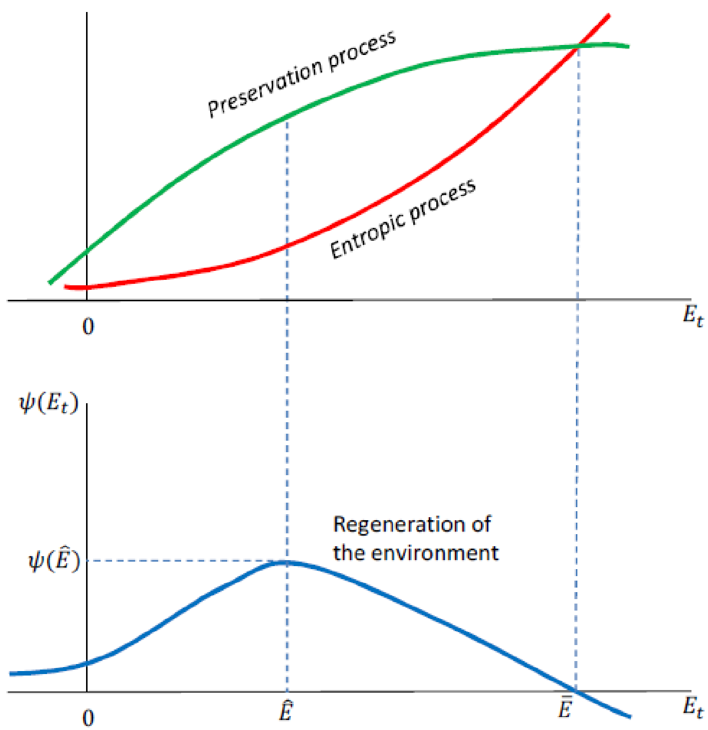

Fig 3. Ecological processes that determine regeneration of the environment

We assume that

[9]For example, Acemoglu et al. (2012) assume that the regeneration of the environment depends positively and linearly on the environmental quality and there is an upper bound for the environment, while Jone and Pecchenino (1994) assume a positive degradation rate of the environment. One can find the alternatives of modeling the dynamics of environment or pollution stock in growth models in Xepapadeas (2005).

$$\begin{cases} \psi'(E) > (<)(=)\, 0 \text{ if } E < (>)(=)\, \hat{E};\ \psi'(E) > -1 \\ \psi''(E) < 0\ \ \forall E \in [0, \bar{E}];\ \text{ and} \\ \lim_{E \to -\infty} \psi(E) = 0;\ \psi(\bar{E}) = 0;\ \psi(E) < 0\ \forall E > \bar{E} \end{cases} \qquad (A1)$$

where $\hat{E} > 0$ is the environmental quality at which the regeneration of environment is maximum; $\bar{E}$ is the highest quality that the environment can reach without human intervention.

The assumption $\psi'(E) > -1$ guarantees that for all E_t^1 and E_t^2 such that $E_t^1 < E_t^2$ then $E_t^1 + \psi(E_t^1) < E_t^2 + \psi(E_t^2)$, i.e. the higher environmental quality is today without any intervention of human, the higher is will be tomorrow. This assumption also guarantees the global stability of the "natural" steady state of environmental quality $\bar{E}$.

3.4 Agents

Each agent lives for two periods. In the first period of life, an agent is endowed with one unit of labor that he supplies inelastically to the labor market and produces intermediate inputs to seek monopoly profits. He divides his income I_t, which is determined in (13), between consumption when young c_{yt} and savings s_t lent to the firms for a rental rate r_{t+1} to be used in period $t+1$ as capital. The gross return on savings is used up as consumption c_{ot+1} when old. The lifetime utility of agents coming from their consumption when young and old and their longevity is as follows

$$u_t = \ln c_{yt} + \phi(E_t) \ln c_{ot+1}$$

where $\phi(E_t) \in [0, 1]$ is the life expectancy of the old agent born at t, depending on the environmental quality E_t.[10]

Since $\phi(E_t)$ also measures the probability for the agent to live through the entire old-age period, then in this paper, we call $\phi(E_t)$ as "life expectancy", or "longevity", or "survival probability" interchangeably. We assume some essential properties of $\phi(E)$ such as

$$\begin{cases} \phi'(E) > 0,\ \phi''(E) < 0,\ \forall E > 0,\ \text{and}\ \phi(E) = 0\ \forall E \leq 0 \\ \lim_{E \to 0^+} \phi'(E) < +\infty;\ \lim_{E \to +\infty} \phi'(E) = 0,\ \lim_{E \to +\infty} \phi(E) \in (0, 1] \end{cases} \qquad (A2)$$

[10] In fact, there is a sizeable literature suggesting that life expectancy also depends on other factors rather than soly environmental quality, such as private and/or public health expenditures, which directly depends on income per capita (Chakraborty 2004, Zhang et al. 2006, Bhattacharya and Qiao 2007, Palivos and Varvarigos 2010, and others). Other papers assume that income per capita has a positive externality on the life expectancy of the old agent (Varvarigos 2010, Goenka et al. 2012, and others). However, we can extend our model by introducing the positive externality of output (or income) per capita on the health profile of the agent without changing the qualitative analysis of the competitive equilibrium and dynamics as follows

$$\tilde{u}_t = \varphi(\bar{y}_t) \left[\ln c_{yt} + \phi(E_t) \ln c_{ot+1}\right]$$

where $\varphi(\bar{y}_t) \in (0, 1]$ measures the externality of output per capita $\bar{y}_t$ in period t on the health profile of the representative agent born at t, and $\varphi'(\bar{y}_t) > 0$.

Note that, for the utility function $\tilde{u}_t$, the output per capita affects both periods of the agent's life. When young, the health profile of the agent is associated with his young consumption and when old, his health profile is transmitted to his life expectancy. Hence, the agent's life expectancy is now $\varphi(\bar{y}_t)\phi(E_t)$. However, the model with this utility function gives us the optimal allocation between the young consumption and savings of the agent the same ones determined in (18) and (19), and crucially, it does not change the properties of the competitive equilibrium, dynamics, and steady states as the present model in this paper provides. The proof for the above statements in this note are fairly straightforward. Moreover, one of the most important focuses in this paper is the impact of environmental quality on physical accumulation. Therefore, for the sake of simplification and without posing an unnecessary cumbersome problem for the social planner, we assume that the life expectancy of the old agent depends solely on environemtal quality. Such an assumption was also recently employed by Mariani et al. (2010).

The lifetime utility maximization problem of the agent t is

$$\max_{c_{yt}, s_t, c^o_{t+1}} \ln c_{yt} + \phi(E_t) \ln c_{ot+1} \tag{15}$$

subject to

$$c_{yt} + s_t = I_t \tag{16}$$

$$c_{ot+1} = \frac{r_{t+1}}{\phi(E_t)} s_t \tag{17}$$

for given I_t, E_t, and r_{t+1}. Because an average fraction $1 - \phi(E_t)$ of the generation born in t cannot live in $t+1$, then the aggregate return on savings is distributed to the survivors of this generation. Therefore, the return on period t savings for the survivors is $\frac{r_{t+1}}{\phi(E_t)}$.

The optimal choices are

$$c_{yt} = \frac{1}{1+\phi(E_t)} I_t \tag{18}$$

$$s_t = \frac{\phi(E_t)}{1+\phi(E_t)} I_t \tag{19}$$

$$c_{ot+1} = \frac{r_{t+1}}{1+\phi(E_t)} I_t \tag{20}$$

From (18) and (19) we find that the allocation between consumption when young and savings depends on the agent's life expectancy. The longer the agent expects to live when old, the less share of income he allocates to consume when young in order to save more for higher consumption when old. That is to say, life expectancy has a positive impact on capital accumulation.

4 Equilibria and dynamics

The competitive equilibrium in the economy set up above is characterized by: (i) the agent's utility maximization (15) under budget constraints (16) and (17); (ii) the law of motion of physical capital $k_{t+1} = s_t$; (iii) the final good producing firm's profit maximization determining the factor prices (2) and (3); (iv) the monopoly profit maximization of the intermediate sector $i \in \{c, d\}$; and (v) the dynamics of the environment. Therefore, a competitive equilibrium allocation $\{c_{yt}, c_{ot+1}, k_{ct+1}, k_{dt+1}, E_t\}_t$, which can fully characterize the competitive equilibrium of the economy, is the solution to the following system of equations:

$$c_{yt} = \frac{1}{1+\phi(E_t)} (1-\alpha^2) A^{1-\alpha} (k_{ct} + k_{dt})^\alpha \tag{21}$$

$$\phi(E_t) c_{ot+1} = \alpha^2 A^{1-\alpha} (k_{ct+1} + k_{dt+1})^\alpha \tag{22}$$

$$k_{ct+1} + k_{dt+1} = \frac{\phi(E_t)}{1+\phi(E_t)} (1-\alpha^2) A^{1-\alpha} (k_{ct} + k_{dt})^\alpha \tag{23}$$

$$E_t = E_{t-1} + \psi(E_{t-1}) - \xi \frac{A_d (k_{ct} + k_{dt})}{A} \tag{24}$$

$$\frac{k_{ct+1}}{A_c} = \frac{k_{dt+1}}{A_d} \tag{25}$$

for given k_{ct},k_{dt}, and E_{t-1}.

The feasibility of the allocation of resources is guaranteed by equations (21), (22), and (23), since in any period t, we have

$$c_{yt} + \phi(E_{t-1})c_{ot} + k_{ct+1} + k_{dt+1} = (1-\alpha^2)A^{1-\alpha}(k_{ct}+k_{dt})^\alpha + \alpha^2 A^{1-\alpha}(k_{ct}+k_{dt})^\alpha = Y_t \tag{26}$$

as stated in (11).

The competitive equilibrium of this economy can be fully characterized by the following reduced system which represents the equilibrium dynamics of physical capital k_{t+1} and of the environment E_{t+1}, as well as the allocation rule of physical capital for the intermediate sectors, i.e.

$$k_{t+1} = \frac{\phi(E_t)(1-\alpha^2)}{1+\phi(E_t)} A^{1-\alpha} k_t^\alpha \tag{27}$$

$$E_{t+1} = E_t + \psi(E_t) - \xi A_d \frac{\phi(E_t)(1-\alpha^2)}{1+\phi(E_t)} A^{-\alpha} k_t^\alpha \tag{28}$$

for given $E_0, k_0 > 0$, where $k_t = k_{ct} + k_{dt}$.

To study the dynamics of this economy, it is sufficient to examine the dynamic system of two equations (27) and (28). In this benchmark model, for a convenience of analyzing the dynamics and number of steady states, we assume that[11]

$$\phi'''(E) = 0 \qquad \text{and} \qquad \frac{\partial^2 \left(\psi'(E)/\psi(E)\right)}{\partial E^2} < 0 \qquad (A3)$$

Hereafter, to lighten notation, we sometimes denote $\phi = \phi(E_t)$, $\psi = \psi(E_t)$, $\phi' = \phi'(E_t)$, $\psi' = \psi'(E_t)$, $\phi'' = \phi''(E_t)$, and so on. Now, we define two important loci, KK and EE.

The KK locus: Let KK be the locus of all $(k_t, E_t) \in \mathbb{R}_+^2$ such that the per capita physical capital is in a steady state:

$$KK \equiv \left\{(k_t, E_t) \in \mathbb{R}_+^2 : k_{t+1} = k_t\right\} \tag{29}$$

$$\text{i.e.,} \qquad k_t = 0 \quad \forall t \qquad \text{or} \qquad k_t = \left[\frac{\phi(E_t)(1-\alpha^2)}{1+\phi(E_t)}\right]^{\frac{1}{1-\alpha}} A \equiv \Omega(E_t) \qquad (KK)$$

Lemma 1: *Under assumptions (A2) and (A3), the function $\Omega(E_t)$ is monotonically increasing in $E_t \in (0, +\infty)$, and there exists a unique $E^p > 0$ such that $\Omega(E_t)$ is strictly convex in $E_t \in (0, E^p)$ and strictly concave in $E_t \in (E^p, +\infty)$. Moreover, $\Omega(E_t)$ is upper bounded.*

Proof: In effect, for $E_t > 0$ we have

$$\Omega'(E_t) = \left[\frac{\phi(1-\alpha^2)}{1+\phi}\right]^{\frac{\alpha}{1-\alpha}} \frac{(1+\alpha)\phi'}{(1+\phi)^2} A > 0$$

i.e. $\Omega(E_t)$ is monotonically increasing in $E_t \in (0, +\infty)$.

[11] Indeed, the assumption $(A3)$ relates to the third derivatives of $\phi(E)$ and $\psi(E)$, which lack background on their signs. The assumption $(A3)$ will be relaxed later for more general analyses.

The second derivative of $\Omega(E_t)$ is

$$\Omega''(E_t) = \left(\phi'^2\left[\frac{\alpha}{(1-\alpha)\phi} - 2\right] + \phi''(1+\phi)\right)\left[\frac{\phi(1-\alpha^2)}{1+\phi}\right]^{\frac{1}{1-\alpha}} \frac{(1+\alpha)A}{(1+\phi)^3}$$

whose sign is that of the function

$$\Lambda(E_t) = \phi'^2\left[\frac{\alpha}{(1-\alpha)\phi} - 2\right] + \phi''(1+\phi)$$

We now prove that there is a unique $E^p > 0$ such that $\Lambda(E^p) = 0$. Indeed, under assumption $(A2)$ we have

$$\lim_{E_t \to 0^+} \Lambda(E_t) = +\infty \qquad \text{and} \qquad \lim_{E_t \to +\infty} \Lambda(E_t) < 0.$$

Hence, there exists $E^p > 0$ such that $\Lambda(E^p) = 0$.

Next, we prove the uniqueness of E^p by a contradiction. Suppose that there were $E^q > 0$ and $E^q \neq E^p$ such that $\Lambda(E^q) = 0$. Without loss of any generality, suppose also that $E^q > E^p$. Since $\phi(E)$, $\phi'(E) > 0$ and $\phi''(E) < 0$ for $E > 0$, then if $\Lambda(E^p) = \Lambda(E^q) = 0$, it must hold that

$$\frac{\alpha}{(1-\alpha)\phi(E^p)} - 2 > \frac{\alpha}{(1-\alpha)\phi(E^q)} - 2 > 0$$

And $\phi'''(E) = 0 \ \forall E > 0$ implies $\phi''(E^p) = \phi''(E^q)$. Therefore, we have

$$\begin{aligned}\Lambda(E^p) &= \phi'(E^p)^2\left[\frac{\alpha}{(1-\alpha)\phi(E^p)} - 2\right] + \phi''(E^p)[1+\phi(E^p)] \\ &> \phi'(E^q)^2\left[\frac{\alpha}{(1-\alpha)\phi(E^q)} - 2\right] + \phi''(E^q)[1+\phi(E^q)] = \Lambda(E^q)\end{aligned}$$

which contradicts $\Lambda(E^p) = \Lambda(E^q) = 0$. Hence, there is a unique $E^p > 0$ such that $\Lambda(E^p) = 0$, and $\Lambda(E_t) > 0$ if $E_t \in (0, E^p)$ and $\Lambda(E_t) < 0$ if $E_t \in (E^p, +\infty)$. That is to say, $\Omega(E_t)$ is strictly convex in $E_t \in (0, E^p)$ and strictly concave in $E_t \in (E^p, +\infty)$.

Moreover, we have

$$\lim_{E_t \to +\infty} \Omega(E_t) = \left[\frac{\bar{\phi}(1-\alpha^2)}{1+\bar{\phi}}\right]^{\frac{1}{1-\alpha}} A$$

which implies that $\Omega(E_t)$ is upper bounded. Q.E.D.

From Lemma 1, the KK locus is depicted by the following figure.

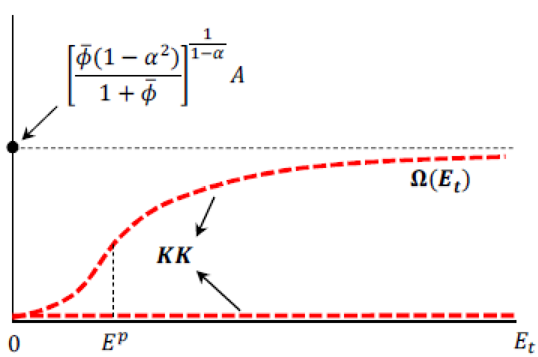

Fig 4. The KK locus

The EE locus: Let EE be the set of all $(k_t, E_t) \in \mathbb{R}_+^2$ such that the environmental quality is in a steady state:

$$EE \equiv \left\{(k_t, E_t) \in \mathbb{R}_+^2 : E_{t+1} = E_t\right\} \tag{30}$$

i.e.

$$k_t = \left[\frac{\psi(E_t)\,[1+\phi(E_t)]}{\phi(E_t)(1-\alpha^2)\xi A_d}\right]^{\frac{1}{\alpha}} A \equiv \Phi(E_t) \qquad (EE)$$

We have

$$\Phi'(E_t) = \frac{A\,[\psi(1+\phi)]^{\frac{1-\alpha}{\alpha}}}{\alpha\,[(1-\alpha^2)\xi A_d]^{\frac{1}{\alpha}}\,\phi^{\frac{1+\alpha}{\alpha}}}\,[\psi'(1+\phi)\phi - \psi\phi']$$

From the last equation, for $\forall E \in (0, \bar{E})$, the sign of $\Phi'(E_t)$ is the sign of $\psi'(1+\phi)\phi - \psi\phi'$, i.e. the sign of the following function

$$\Theta(E_t) = \frac{\psi'(E_t)}{\psi(E_t)} - \frac{\phi'(E_t)}{[1+\phi(E_t)]\phi(E_t)} \tag{31}$$

i.e. for $E_t \in (0, \bar{E})$ we have

$$\Phi'(E_t) < (=)(>)0 \ \text{ if } \ \Theta(E_t) < (=)(>)0$$

Now we consider the equation implying $\Phi'(E_t) = 0$ for $E_t \in (0, \bar{E})$, i.e.

$$\frac{\psi'}{\psi} = \frac{\phi'}{(1+\phi)\phi} \tag{32}$$

We have,

$$\frac{\partial\,(\psi'/\psi)}{\partial E_t} = \frac{\psi''\psi - \psi'^2}{\psi^2} < 0$$

So, under assumptions $(A1)$ and $(A3)$, the left-hand side of equation (32) is a decreasing and strictly concave function of E_t. We also have

$$\frac{\psi'(0)}{\psi(0)} > 0 \qquad \text{and} \qquad \lim_{E_t \to \bar{E}^-} \frac{\psi'(E_t)}{\psi(E_t)} = -\infty.$$

Under assumptions $(A2)$ and $(A3)$, the right-hand side of equation (32) is a decreasing and strictly convex function of E_t. In effect,

$$\frac{\partial\,(\phi'/[(1+\phi)\phi])}{\partial E_t} = \frac{\phi''(1+\phi)\phi - \phi'^2(1+2\phi)}{[(1+\phi)\phi]^2} < 0$$

and (note that $\phi''' = 0$)

$$\frac{\partial^2\,(\phi'/[(1+\phi)\phi])}{\partial E_t^2} = \frac{2\phi'^3(1+\phi)\phi - (1+2\phi)\phi'\,[3\phi''(1+\phi)\phi - 2\phi'^2(1+2\phi)]}{[(1+\phi)\phi]^3} > 0$$

We also have

$$\lim_{E_t \to 0^+} \frac{\phi'(E_t)}{[1+\phi(E_t)]\phi(E_t)} = +\infty \qquad \text{and} \qquad \frac{\phi'(\bar{E})}{[1+\phi(\bar{E})]\phi(\bar{E})} > 0.$$

Lemma 2: *Under assumptions (A1), (A2), and (A3), the number of solutions* $E \in [0, \bar{E}]$ *to equation* (32) *is less than or equal to* 2, *i.e.*

$$0 \leq n = \left\| \left\{ E \in [0, \bar{E}] : \ \Theta(E) = \frac{\psi'(E)}{\psi(E)} - \frac{\phi'(E)}{[1 + \phi(E)]\phi(E)} = 0 \right\} \right\| \leq 2$$

where $\|\cdot\|$ *is the cardinal of a set.*

Proof: It is straightforward from the strict concavity of $\frac{\psi'(E)}{\psi(E)}$ and strict convexity of $\frac{\phi'(E)}{[1+\phi(E)]\phi(E)}$.

Remarks: Under assumptions $(A1)$, $(A2)$, and $(A3)$:

(i) If $n = 0$ then $\Phi'(E_t) < 0 \ \forall E_t \in (0, \bar{E})$, if $n = 1$ then $\Phi'(E_t) \leq 0 \ \forall E_t \in (0, \bar{E}]$, and in both cases $\Phi(E_t)$ is monotonically decreasing in $E_t \in (0, \bar{E}]$.

(ii) If $n = 2$ then $\Phi(E_t)$ has one local minimum and one local maximum.

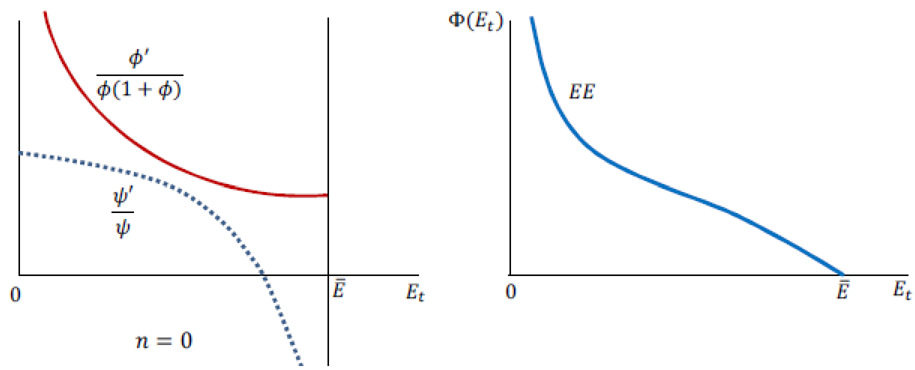

Fig 5a. Monotonic decreasing locus EE

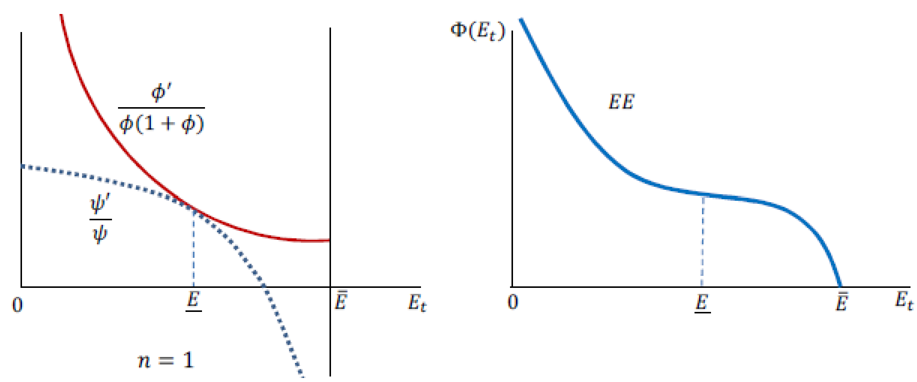

Fig 5b. Monotonic decreasing locus EE

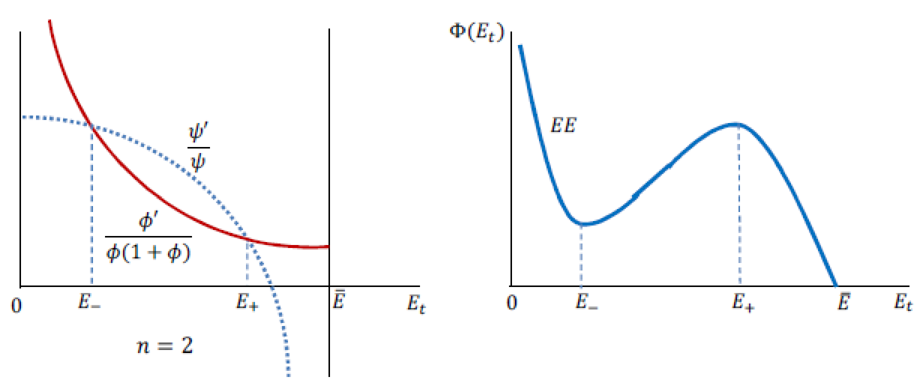

Fig 5c. Locus EE with one local minimum and one local maximum

Lemma 3: *For the dynamics system* $(k_t, E_t)_t$ *characterized by equations* (27)-(28)*, then:*

$$\text{(i)} \quad k_{t+1} - k_t \begin{cases} > 0 & \text{if} \quad 0 < k_t < \Omega(E_t) \\ = 0 & \text{if} \quad k_t = \Omega(E_t) \\ < 0 & \text{if} \quad k_t > \Omega(E_t) \end{cases} \quad \textit{and} \quad \text{(ii)} \quad E_{t+1} - E_t \begin{cases} > 0 & \text{if} \quad k_t < \Phi(E_t) \\ = 0 & \text{if} \quad k_t = \Phi(E_t) \\ < 0 & \text{if} \quad k_t > \Phi(E_t) \end{cases}$$

Proof: The proof of this Lemma is fairly straightforward. Q.E.D.

Under Lemma 1, Lemma 2, and Lemma 3, the dynamics of the economy set up above can be represented by one of the following diagrams.

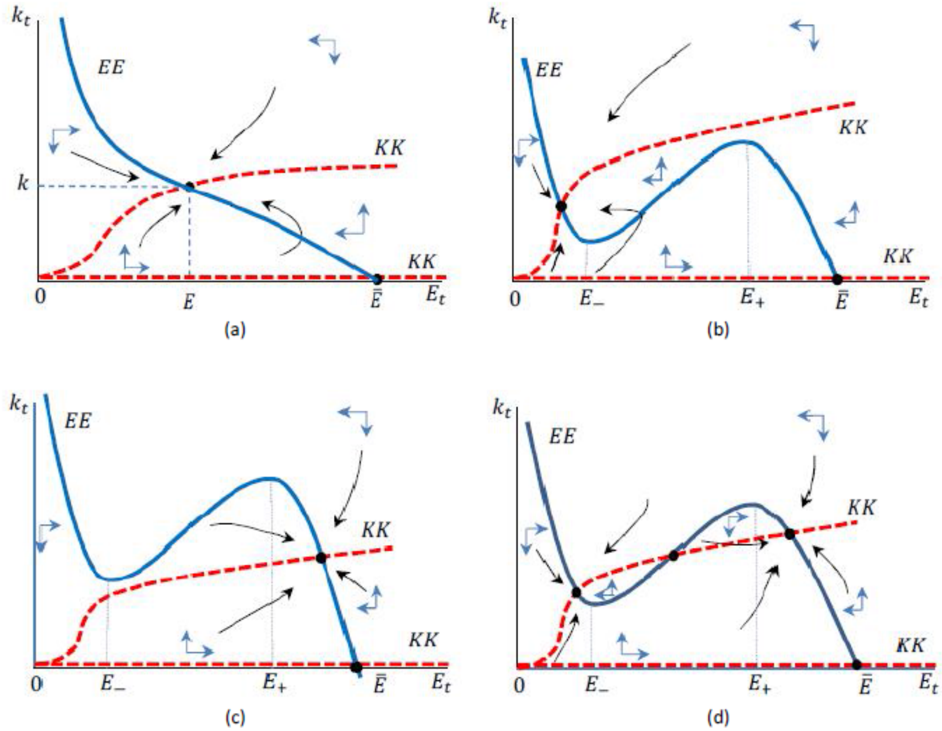

Fig 6. The dynamics

From the dynamical system (27) and (28), as well as the diagrams above, there is always a trivial steady state characterized by $(E, k) = (\bar{E}, 0)$. This competitive steady state prevails only when there is no production in the economy (i.e. there is no pollution), and in this case the environment converges to the size of "the garden of Eden", $\bar{E}$, at which the entropic process is completely offset by the natural preservation process.

5 Competitive steady states and their stability

5.1 Interior competitive steady state

Analytically, from the competitive equilibrium characterized by the system of equations from (21) to (25), the steady states (c_y, c_o, k_c, k_d, E) are characterized by

$$c_y = \frac{(1-\alpha^2)A^{1-\alpha}(k_c + k_d)^\alpha}{1 + \phi(E)} \tag{33}$$

$$c_o = \frac{\alpha^2 A^{1-\alpha}(k_c + k_d)^\alpha}{\phi(E)} \tag{34}$$

$$k_c + k_d = \left[\frac{\phi(E)(1-\alpha^2)}{1+\phi(E)}\right]^{\frac{1}{1-\alpha}} A \tag{35}$$

$$\psi(E) = \xi k_d \tag{36}$$

$$\frac{k_c}{A_c} = \frac{k_d}{A_d} \tag{37}$$

Formally, the competitive steady states can be fully determined by

$$\psi(E) = \xi A_d \left[\frac{\phi(E)(1-\alpha^2)}{1+\phi(E)}\right]^{\frac{1}{1-\alpha}} \tag{38}$$

Now, it is interesting to clarify the exact cases in which the multiple steady states or unique steady state prevail. We rewrite equation (38) as

$$\psi(E) \left[\frac{1+\phi(E)}{\phi(E)(1-\alpha^2)}\right]^{\frac{1}{1-\alpha}} = \xi A_d \tag{39}$$

where the right-hand side of (39) is constant.

Since the left-hand side of (39) is a function of E, then let us define it as $G(E)$. We have

$$G'(E) = \frac{\psi(E)}{1-\alpha} \left[\frac{1+\phi(E)}{\phi(E)(1-\alpha^2)}\right]^{\frac{1}{1-\alpha}} \left[(1-\alpha)\frac{\psi'(E)}{\psi(E)} - \frac{\phi'(E)}{[1+\phi(E)]\phi(E)}\right]$$

The sign of $G'(E)$ is that of the function

$$\hat{\Theta}(E) = (1-\alpha)\frac{\psi'(E)}{\psi(E)} - \frac{\phi'(E)}{[1+\phi(E)]\phi(E)} \tag{40}$$

Hence, for $E \in (0, \bar{E})$, we have

$$G'(E) > (=)(<) 0 \quad \text{if} \quad \hat{\Theta}(E) > (=)(<) 0 \tag{41}$$

We also have

$$\lim_{E \to 0^+} G(E) = +\infty \qquad \text{and} \qquad \lim_{E \to \bar{E}^-} G(E) = 0$$

Note that function $\hat{\Theta}(E)$ in (40) only differs from the function $\Theta(E)$ in (31) by the term $(1-\alpha)$. For any $\alpha \in (0,1)$, assumptions $(A1)$ and $(A3)$ still guarantee the strict concavity of $(1-\alpha)\frac{\psi'(E)}{\psi(E)}$. Recall that the strict convexity of $\frac{\phi'(E)}{\phi(E)[1+\phi(E)]}$ is guaranteed by assumption $(A2)$ as clarified before. Now, let us define the set

$$S \equiv \left\{ E \in (0, \bar{E}) : \ \hat{\Theta}(E) = 0 \right\}$$

So, similar to the statement in Lemma 2, under $(A1)$, $(A2)$, and $(A3)$, we have $\|S\| \leq 2$. If $\|S\| = 2$ then $G(E)$ has one local minimum E_m, and one local maximum E_M, in which $0 < E_m < E_M < \bar{E}$. The number of interior steady states is stated in Proposition 1.

Proposition 1: *In the overlapping generations economy with the environmental externality set up above, under assumptions (A1), (A2), and (A3):*

(i) *In the case $\|S\| \leq 1$, there always exists only a unique interior steady state $(E, k) \gg (0, 0)$.*

(ii) *In the case $\|S\| = 2$,*

(ii.a) *If the dirty intermediate sector is too polluted and/or its productivity is rather high, specifically $\xi A_d > G(E_M)$, then there is an interior steady state characterized by $E < E_m$. If $\xi A_d = G(E_M)$, there exists an another interior steady state characterized by $E = E_M$.*

(ii.b) *If the dirty intermediate sector is rather clean and/or its productivity is too low, specifically $\xi A_d < G(E_m)$ then there is an interior steady state characterized by $E > E_M$. If $\xi A_d = G(E_m)$, there exists an another interior steady state characterized by $E = E_m$.*

(ii.c) *If $G(E_m) < \xi A_d < G(E_M)$, then there exists three distinct steady states $E_l < E_u < E_h$. Moreover, the following properties hold, $E_l < E_m < E_u < E_M < E_h$.*

Proof: These statements about the existence of steady states in Proposition 1 can be proven fairly straightforward by following graphs.

(i) In the case $\|S\| \leq 1$

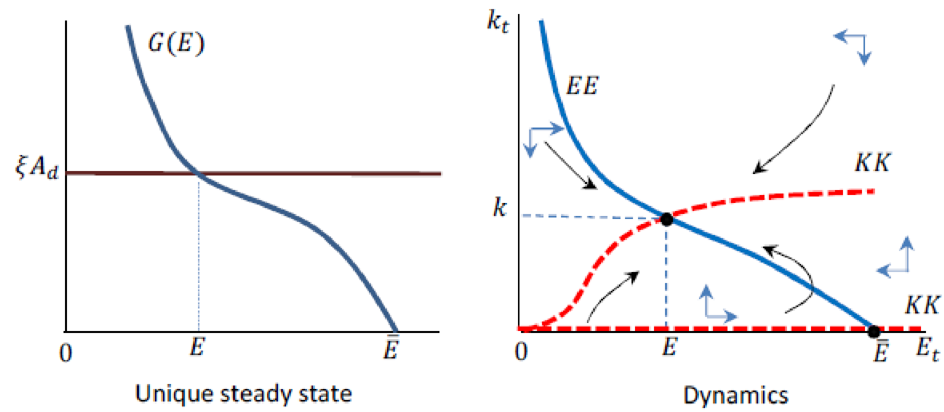

Fig 7a. Unique interior steady state and dynamics.

(ii) In the case $\|S\| = 2$

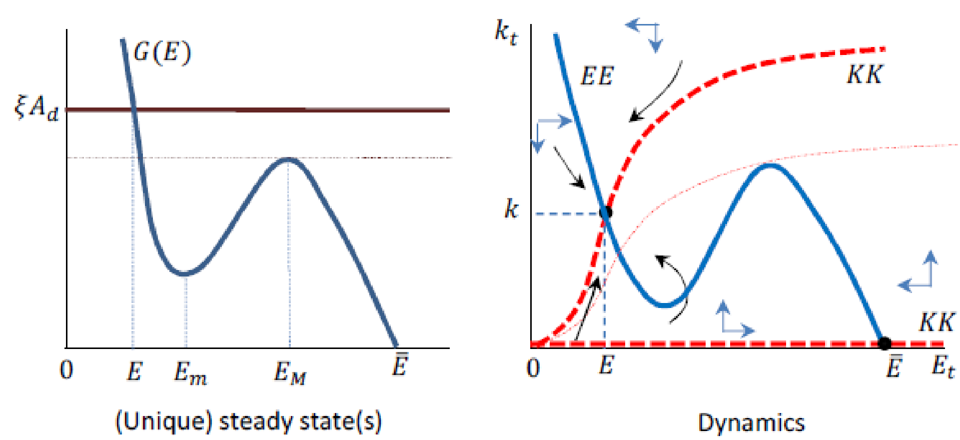

Fig 7b. (Unique) interior steady state(s) and dynamics in the case $\xi A_d(1-\alpha^2)^{\frac{1}{1-\alpha}} \geq G(E_M)$.

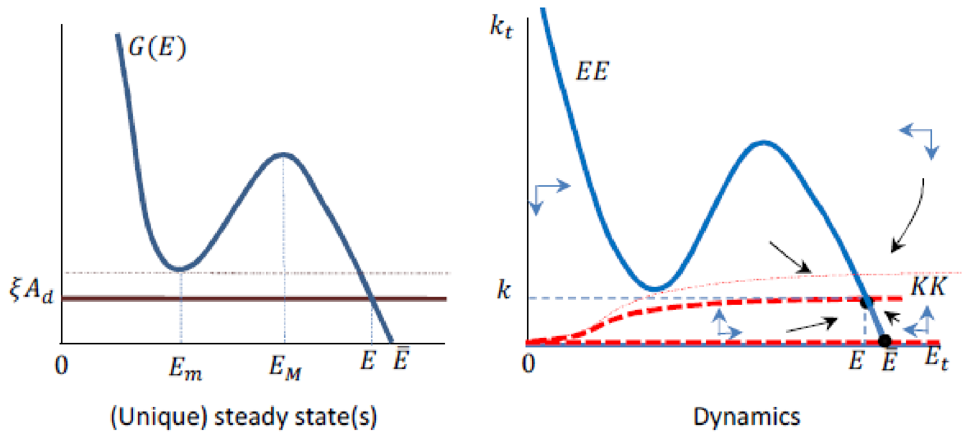

Fig 7c. (Unique) interior steady state(s) and dynamics in the case $\xi A_d \le G(E_m)$.

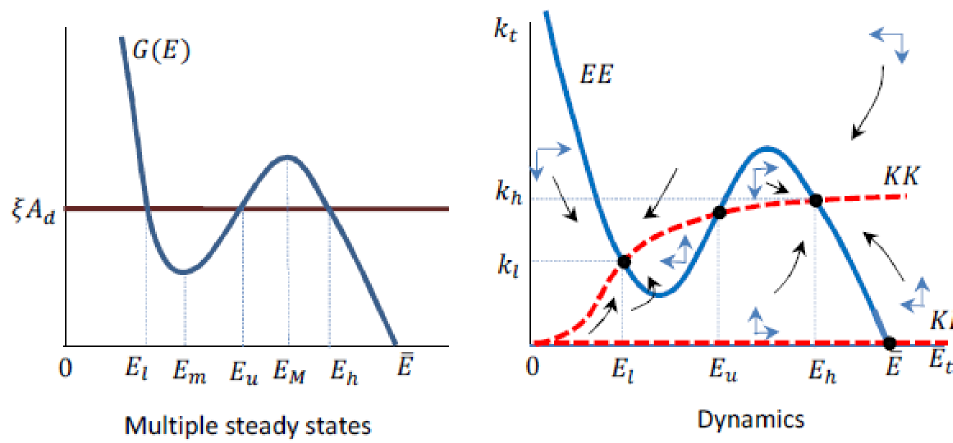

Fig 7d. Multiple interior steady states and dynamics in the case $G(E_m) < \xi A_d < G(E_M)$.

Q.E.D.

If the case that $\|S\| \le 1$ prevails, we always have a unique interior steady state no matter how polluted, $\xi A_d > 0$, the dirty intermediate sector is. It is straightforward that

$$\lim_{\alpha \to 1^-} \|S\| = 0$$

which implies that there always exists $\tilde{\alpha} \in (0, 1)$ such that $\|S\| \le 1$ $\forall \alpha \in [\tilde{\alpha}, 1)$ and $\|S\| = 0$ $\forall \alpha \in (\tilde{\alpha}, 1)$. So, the case $\|S\| \le 1$ occurs when at least one of the following two conditions occurs: (i) $\frac{\psi'(E)}{\psi(E)} \le \frac{\phi'(E)}{\phi(E)[1+\phi(E)]}$, and/or (ii) the elasticity of the final output to the intermediate input α is too high, i.e. $\alpha \in [\tilde{\alpha}, 1)$.

If the case $\|S\| = 2$ prevails, there may be multiple interior steady states. The following Lemma reveals some important properties.

Lemma 4: *In the overlapping generations economy with the environmental externality set up above, and in the case multiple interior steady states prevails, i.e.* $G(E_m) < \xi A_d < G(E_M)$, *the following properties hold:*

(i) $E_M < \hat{E}$; *and*

(ii) *The number of interior steady states characterized by* $E \ge \hat{E}$ *is at most one.*

***Proof*:** (i) We prove this property by supposing a negation that $E_M \ge \hat{E}$, then

$$\hat{\Theta}(E_M) = (1-\alpha)\frac{\psi'(E_M)}{\psi(E_M)} - \frac{\phi'(E_M)}{[1+\phi(E_M)]\phi(E_M)} < 0$$

which contradicts the property that $\hat{\Theta}(E_M) = 0$. Hence, $E_M < \hat{E}$.

(ii) Suppose that there are at least two interior steady states that satisfy$E \geq \hat{E}$. That is to say, $E_h > E_u \geq \hat{E}$. We know that $E_u < E_M \Rightarrow E_M > \hat{E}$, which contradicts the property proven in (i). Q.E.D.

We now consider the cases that the dirty intermediate sector is polluted enough to guarantee the existence of a low steady state characterized by low environmental quality and low physical capital per capita; specifically, $\xi A_d > G(E_m)$. The welfare property of interior steady states in this case are stated in the following proposition.

Proposition 2: *In the overlapping generations economy with the environmental externality set up above, in the case $\|S\| = 2$ and $\xi A_d > G(E_m)$:*

(i) *The steady state associated with higher environmental quality has the higher stationary regeneration of the environment.*

(ii) *For a given level of aggregate total factor productivity $A = A_c + A_d$, if the elasticity of final output to intermediate inputs, α, is too low, then a competitive interior steady state associated with higher environmental quality provides a lower utility to the agent.*

(iii) *For a given elasticity of final output to intermediate inputs $\alpha > 0$, and the case of the multiple steady states prevails, i.e. $G(E_M) > \xi A_d > G(E_m)$, then there always exists a level of aggregate total factor productivity $A = A_c + A_d$ that is high enough such that a competitive interior steady state associated with higher environmental quality provides higher utility to the agent.*

Proof: (i) At a steady state equilibrium, it holds

$$\psi(E) = \xi A_d \left[\frac{\phi(E)(1-\alpha^2)}{1+\phi(E)}\right]^{\frac{1}{1-\alpha}}$$

where the right-hand side of the last equation is increasing in E. That is to say, the higher stationary of environmental quality, the higher the environment's stationary regeneration.

(ii) At an interior steady state equilibrium we have

$$c_y = \frac{(1-\alpha^2)A^{1-\alpha}k^\alpha}{1+\phi(E)} = \frac{(1-\alpha^2)A}{(\xi A_d)^\alpha}\frac{\psi(E)^\alpha}{1+\phi(E)} \quad \text{and} \quad c_o = \frac{\alpha^2 A^{1-\alpha}k^\alpha}{\phi(E)} = \frac{\alpha^2 A}{(\xi A_d)^\alpha}\frac{\psi(E)^\alpha}{\phi(E)}$$

Hence the utility at a steady state equilibrium is

$$u = \ln\left[\frac{(1-\alpha^2)A}{(\xi A_d)^\alpha}\frac{\psi(E)^\alpha}{1+\phi(E)}\right] + \phi(E)\ln\left[\frac{\alpha^2 A}{(\xi A_d)^\alpha}\frac{\psi(E)^\alpha}{\phi(E)}\right] \equiv V(E)$$

We have

$$V'(E) = \frac{\alpha\psi'(E)}{\psi(E)}[1+\phi(E)] + \phi'(E)\left(\ln\left[\frac{\alpha^2 A}{(\xi A_d)^\alpha}\frac{\psi(E)^\alpha}{\phi(E)}\right] - \frac{2+\phi(E)}{1+\phi(E)}\right) \tag{42}$$

So, for any given level of aggregate productivity A,

$$\lim_{\alpha\to 0^+} V'(E) = -\infty$$

which implies that $\exists\hat{\alpha} > 0$ such that $\forall\alpha \in (0, \hat{\alpha})$, it holds that $V'(E) < 0$.

(iii) Firstly, we consider $V'(E)$ in the interval $E \in (0, \hat{E}]$. In this interval we have $\psi'(E) \geq 0$. Since, $A_d < A$ then from (42) we have

$$V'(E) > \frac{\alpha\psi'(E)}{\psi(E)}[1+\phi(E)] + \phi'(E)\left(\ln\frac{\alpha^2 A^{1-\alpha}}{\xi^\alpha} + \ln\frac{\psi(E)^\alpha}{\phi(E)} - \frac{2+\phi(E)}{1+\phi(E)}\right)$$

So, it is straightforward that $\forall E \in (0, \hat{E}]$ there always exists A high enough to guarantee

$$V'(E)|_{E\in(0,\hat{E}]} > 0 \tag{43}$$

Hence, if $E_h \leq \hat{E}$ then we have from (43) that $V(E_l) < V(E_u) < V(E_h)$.

As stated in Lemma 4, the number of interior steady states characterized by $E \geq \hat{E}$ is at most one, therefore, $E_u < \hat{E}$. Now, if $E_h > \hat{E}$ then $\exists E_h^- < \hat{E}$ and $\psi(E_h^-) = \psi(E_h) > \psi(E_u)$, so $E_h^- > E_u$. Since $E_h^- \in (E_u, \hat{E})$ then as proven above, there exists A high enough such that $V(E_h^-) > V(E_u)$. Now we prove that there also always exists A high enough such that $V(E_h) > V(E_h^-)$. In effect,

$$V(E_h) - V(E_h^-) = \ln\frac{1+\phi(E_h^-)}{1+\phi(E_h)} + \phi(E_h)\ln\left[\frac{\alpha^2 A}{(\xi A_d)^\alpha}\frac{\psi(E_h)^\alpha}{\phi(E_h)}\right] - \phi(E_h^-)\ln\left[\frac{\alpha^2 A}{(\xi A_d)^\alpha}\frac{\psi(E_h^-)^\alpha}{\phi(E_h^-)}\right]$$

$$> \ln\frac{1+\phi(E_h^-)}{1+\phi(E_h)} + [\phi(E_h) - \phi(E_h^-)]\ln\frac{\alpha^2 A^{1-\alpha}}{\xi^\alpha} + \phi(E_h)\ln\frac{\psi(E_h)^\alpha}{\phi(E_h)} - \phi(E_h^-)\ln\frac{\psi(E_h^-)^\alpha}{\phi(E_h^-)}$$

Hence, there exits $A > 0$ high enough to guarantee $V(E_h) - V(E_h^-) > 0$. In summary, there always exists A high enough such that $V(E_l) < V(E_u) < V(E_h)$. Q.E.D.

5.2 Stability/Unstability of the steady states

In order to evaluate the stability/unstability of the steady states, we linearize the dynamic system (27) and (28) around each steady state (E, k):

$$\begin{pmatrix} k_{t+1} - k \\ \\ E_{t+1} - E \end{pmatrix} \simeq \underbrace{\begin{pmatrix} \alpha & \frac{\phi'(E)(1-\alpha^2)}{[1+\phi(E)]^2} A \left[\frac{\phi(E)(1-\alpha^2)}{1+\phi(E)}\right]^{\frac{\alpha}{1-\alpha}} \\ \\ -\frac{\alpha}{A}\psi(E)\left[\frac{1+\phi(E)}{\phi(E)(1-\alpha^2)}\right]^{\frac{1}{1-\alpha}} & 1 + \psi'(E) - \frac{\psi(E)\phi'(E)}{\phi(E)[1+\phi(E)]} \end{pmatrix}}_{J} \begin{pmatrix} k_t - k \\ \\ E_t - E \end{pmatrix}$$

where the determinant and trace of associated Jacobian matrix J are

$$\det(J) = \alpha\,[1 + \psi'(E)] > 0$$

$$Tr(J) = \alpha + 1 + \psi'(E) - \frac{\psi(E)\phi'(E)}{\phi(E)[1+\phi(E)]} = \det(J) + 1 + \hat{\Theta}(E)\psi(E)$$

where $\hat{\Theta}(E) = (1-\alpha)\frac{\psi'(E)}{\psi(E)} - \frac{\phi'(E)}{\phi(E)[1+\phi(E)]}$ as defined in the previous section.

Lemma 5:

(i) For the steady state with environmental quality low enough such that $\psi'(E) \geq \frac{1}{\alpha} - 1$*, then such a steady state is not a stable node.*

(ii) For the steady state with environmental quality high enough such that $\psi'(E) < \frac{1}{\alpha} - 1$*, and* $E \leq \hat{E}$*, then:*

(ii.a) *If* $\hat{\Theta}(E) > 0$*, then the steady state is a saddle point.*

(ii.b) *If* $\hat{\Theta}(E) < 0$*, the Jacobian matrix has two eigenvalues with absolute values (if they are real) or with moduli (if they are complex) strictly less than* 1 *(i.e. the steady state is locally stable) if, and only if* $1 + \alpha > \frac{\psi(E)}{2+\psi'(E)} \frac{\phi'(E)}{\phi(E)[1+\phi(E)]}$ *holds at the steady state.*[12]

(iii) (A sufficient condition) For the steady state with environmental quality $E \in (\hat{E}, \bar{E})$*, if* $1 + \alpha \geq \frac{\psi(\hat{E})\phi'(\hat{E})}{\phi(\hat{E})[1+\phi(\hat{E})]}$ *then the steady state is locally stable.*[13]

***Proof*:** See Appendix A2.

Lemma 5 provides us with the stability/unstability properties of the steady states. We find that steady states with too low environmental qualities such that $\psi'(E) \geq \frac{1}{\alpha} - 1$ and the steady states at which $\hat{\Theta}(E) > 0$ are unstable. The steady state at which $\hat{\Theta}(E) < 0$ (including the case of $E \in (\hat{E}, \bar{E})$) and the environmental quality high enough, i.e. $\psi'(E) < \frac{1}{\alpha} - 1$, can be locally stable or unstable depending on the functions $\psi(E)$, $\phi(E)$, and the parameter α. In this paper, we restrict our attention to the interesting class of economies satisfying that if they have a steady state characterized by $E \in (\hat{E}, \bar{E})$, then such a steady state is locally stable. Under this assumption, we will design fiscal policies to help an economy to escape/avoid a so-called poverty (or poverty-environment trap which will be defined in Section 7) and to converge to the first-best steady state (from the viewpoint of the social planner which will be examined in Section 6).

6 The social planner's steady state

This section considers the optimal allocation from a benevolent social planner's viewpoint. The social planner allocates resources so as to maximize the weighted sum of the welfare of current and all future generations. The optimal (in the Pareto sense) allocation chosen by the social planner is the solution $\{c_{yt}, c_{ot+1}, k_{ct+1}, k_{dt+1}, E_t\}_{t=0}^{+\infty}$ to the following problem

$$\max_{\{c_{yt}, c_{ot+1}, k_{ct+1}, k_{dt+1}, E_t\}_{t=0}^{+\infty}} \sum_{t=0}^{+\infty} \frac{\ln c_{yt} + \phi(E_t) \ln c_{ot+1}}{(1+R)^t} \tag{44}$$

subject to, $\forall t = 0, 1, 2, ...$

$$A_c^{1-\alpha} k_{ct}^{\alpha} + A_d^{1-\alpha} k_{dt}^{\alpha} = c_{yt} + \phi(E_{t-1}) c_{ot} + k_{ct+1} + k_{dt+1} \tag{45}$$

$$E_t = E_{t-1} + \psi(E_{t-1}) - \xi k_{dt} \tag{46}$$

[12] In the case $\hat{\Theta}(E) = 0$, the Jacobian matrix has two real distinct eigenvalues $\lambda_1 = \alpha [1 + \psi'(E)]$ and $\lambda_2 = 1$. Because $\lambda_2 = 1$, i.e. the corresponding steady state is non-hyperbolic, the stability type of the steady state cannot be examined on the basis of the eigenvalues. Since the stability of such a steady state is not so important for designing a welfare-improvement tax policy in this paper, then we keep silent for the case $\hat{\Theta}(E) = 0$.

[13] This statement only provides a *sufficient condition* for the steady state with $E \in (\hat{E}, \bar{E})$ to be locally stable. In order to guarantee such a steady state to be locally stable, it is not necessary to assume strictly that such a sufficient condition holds. Indeed, the necessary and sufficient condition for local stability of the steady state in this case is $\{\det(J) > Tr(J) - 1 \text{ and } \det(J) > -Tr(J) - 1\}$, i.e. $1 + \alpha > \frac{\psi(E)}{2+\psi'(E)} \frac{\phi'(E)}{\phi(E)[1+\phi(E)]}$. It is straightforward that, for $E \in (\hat{E}, \bar{E})$, the sufficient condition in this statement is a subset of the necessary and sufficient condition above. That is to say, we have room to assume that the steady state with $E \in (\hat{E}, \bar{E})$ is locally stable.

for given initial conditions c_{o0}, k_{c0}, k_{d0}, and E_{-1}; where $R \geq 0$ is the subjective discount rate of the social planner. The discount rate R is strictly positive when the social planner always cares more about a current generation's welfare than future ones, while R equals zero when she treats all generation equally. To solve this problem, see the Appendix A1.

The social planner's choice at a steady state is $(\bar{c}_y^*, \bar{c}_o^*, \bar{k}_c^*, \bar{k}_d^*, \bar{E}^*)$ which satisfies

$$\frac{\bar{c}_o^*}{\bar{c}_y^*} = 1 + R$$

$$\bar{k}_c^* = \left(\frac{\alpha}{1+R}\right)^{\frac{1}{1-\alpha}} A_c$$

$$\bar{k}_d^* = \left[\frac{\alpha\left[\psi'(\bar{E}^*) - R\right]}{(1+R)\left[\psi'(\bar{E}^*) - R\right] - \phi'(\bar{E}^*)\left(\ln \bar{c}_o^* - 1\right)\xi\bar{c}_o^*}\right]^{\frac{1}{1-\alpha}} A_d$$

$$A_c^{1-\alpha}\bar{k}_c^{*\alpha} + A_d^{1-\alpha}\bar{k}_d^{*\alpha} = \bar{c}_y^* + \phi(\bar{E}^*)\bar{c}_o^* + \bar{k}_c^* + \bar{k}_d^*$$

$$\psi(\bar{E}^*) = \xi\bar{k}_d^*$$

In general, this steady state allocation differs from the competitive one characterized from (33) to (37). The difference between these steady state allocations is not only because of imperfect altruism between generations in the competitive economy but also because individuals cannot internalize the effects of their savings (capital accumulation) and capital allocation on environmental quality through producing dirty intermediate inputs, whereas the social planner can.

If the social planner cares about all generations equally, i.e. $R = 0$,[14] then the social planner's steady state is the so-called golden rule (or the first-best) steady state $(c_y^*, c_o^*, k_c^*, k_d^*, E^*)$ that maximizes the representative agent's utility and is a solution to the following system

$$\frac{c_o^*}{c_y^*} = 1 \tag{47}$$

$$k_c^* = \alpha^{\frac{1}{1-\alpha}} A_c \tag{48}$$

$$k_d^* = \left(\frac{\psi'(E^*)\alpha}{\psi'(E^*) - \phi'(E^*)\left(\ln c_o^* - 1\right)\xi c_o^*}\right)^{\frac{1}{1-\alpha}} A_d \tag{49}$$

$$A_c^{1-\alpha}k_c^{*\alpha} + A_d^{1-\alpha}k_d^{*\alpha} = \left[1 + \phi(E^*)\right]c_o^* + k_c^* + k_d^* \tag{50}$$

$$\psi(E^*) = \xi k_d^* \tag{51}$$

In order to avoid unnecessary cumbersome notations, the analyses in this paper focus on the case $R = 0$ although it can be redone for any $R \in (0, 1)$. Let us define

$$f_c = A_c^{1-\alpha}k_c^\alpha - k_c \equiv f_c(k_c) \tag{52}$$

[14]Note that if $R = 0$ the sum in (44) does not converge. However, as in Gutierrez (2008), we consider here the borderline case ($R = 0$), because we also discuss the optimality of using the *overtaking criterion* mentioned in Burmeister (1980). The idea is that the feasible path A overtakes the feasible path B if there exists a finite $t^* > 0$ such that $\sum_{t=0}^{t^*} \frac{\ln c_{yt} + \phi(E_t)\ln c_{ot+1}}{(1+R_t^A)^t} > \sum_{t=0}^{t^*} \frac{\ln c_{yt} + \phi(E_t)\ln c_{ot+1}}{(1+R_t^B)^t}$ and $\sum_{t=0}^{T} \frac{\ln c_{yt} + \phi(E_t)\ln c_{ot+1}}{(1+R_t^A)^t} > \sum_{t=0}^{T} \frac{\ln c_{yt} + \phi(E_t)\ln c_{ot+1}}{(1+R_t^B)^t}$ for all $T > t^*$, where $R_t^A, R_t^B \geq 0$ are the discount rates of paths A and B respectively. A path is called optimal if it is feasible and overtakes all other feasible paths.

$$f_d = A_d^{1-\alpha} k_d^\alpha - k_d \equiv f_d(k_d) \quad (53)$$

as the net final outputs produced by clean and dirty intermediate inputs, respectively.

From (48) and (52), we find that $f_c^{'}(k_c^*) = 0$, which implies that $f_c^* = f_c(k_c^*)$ gets maximum. That is because the production of clean intermediate inputs does not have any pollution externality affecting the social welfare, so the social planner allocates capital to producing the clean intermediate inputs so as to maximize f_c^* and, ultimately, the social welfare. However, for $f_d^* = f_d(k_d^*)$, from (49) and (53) we have **in general** $f_d^{'}(k_d^*) \neq 0$, which implies that f_d^* does not get maximum because the social planner internalizes the pollution externality resulting from producing the dirty intermediate inputs. Indeed, $f_d^{'}(k_d^*) = 0$ if, and only if $k_d^* = \alpha^{\frac{1}{1-\alpha}} A_d$ which occurs *by chance* only if $\ln c_o^* - 1 = 0$ (i.e. $c_o^* = \mathrm{e}$) and the following condition holds

$$\psi\left(\phi^{-1}\left(\frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} A}{\mathrm{e}} - 1\right)\right) = \xi \alpha^{\frac{1}{1-\alpha}} A_d$$

Now we investigate some important properties of the first-best steady state. We consider the economy under some crucial assumptions

$$\psi(\hat{E}) \geq \xi A_d \left[\alpha \frac{\phi(\bar{E})}{1+\phi(\bar{E})} \frac{1+\phi(E_+^*)}{\phi(E_+^*)}\right]^{\frac{1}{1-\alpha}} \quad (A4)$$

where E_+^* satisfies $\psi(E_+^*) = \xi A_d \alpha^{\frac{1}{1-\alpha}}$ and $\psi'(E_+^*) < 0$.

We also assume that

$$\frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} A}{1+\phi(E_+^*)} \geq \mathrm{e} \quad (A5)$$

Assumption $(A4)$ implies that $\psi(\hat{E}) > \xi A_d \alpha^{\frac{1}{1-\alpha}}$, i.e. the pollution flow under which the net final output produced by the dirty intermediate f_d^* gets maximum is always less than the maximal regeneration capacity of the environment. Assumption $(A5)$ implies that we consider the economies with the elasticity of final output to intermediate inputs is not too high and not too low, and the aggregate productivity A is sufficiently high.

Lemma 6: *Under assumptions (A4) and (A5), these properties hold:*

(i) $k_d^* \leq A_d \alpha^{\frac{1}{1-\alpha}}$*; and*
(ii) $E^* \geq E_+^*$.

Proof: (i) From (49) it is sufficient to prove (i) by showing that $m \leq 1$ where $m = \frac{\psi'(E^*)}{\psi'(E^*) - \phi'(E^*)(\ln c_o^* - 1)\xi c_o^*}$. We prove $m \leq 1$ by a contradiction. Suppose that $m > 1$, then $k_d^* > A_d \alpha^{\frac{1}{1-\alpha}}$ and we prove that the allocation $(c_y^*, c_o^*, k_c^*, k_d^*, E^*)$ would be strictly dominated by the following allocation $(c_y^+, c_o^+, k_c^+, k_d^+, E^+) = (\frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} A}{1+\phi(E_+^*)}, \frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} A}{1+\phi(E_+^*)}, A_c \alpha^{\frac{1}{1-\alpha}}, A_d \alpha^{\frac{1}{1-\alpha}}, E_+^*)$. Note that, the allocation $(c_y^+, c_o^+, k_c^+, k_d^+, E^+)$ is feasible by construction. It is obvious that:

$$f_c(k_c^+) + f_d(k_d^+) = [1+\phi(E^+)]c_o^+ > [1+\phi(E^*)]c_o^* = f_c(k_c^*) + f_d(k_d^*)$$

And because $k_d^* > k_d^+$ then $\psi(E^*) = \xi k_d^* > \xi k_d^+ = \psi(E^+)$, and we defined in assumption $(A5)$ above that $\psi'(E^+) < 0$, therefore $E^* < E^+$. The utilities given by the allocations $(c_y^+, c_o^+, k_c^+, k_d^+, E^+)$ and $(c_y^*, c_o^*, k_c^*, k_d^*, E^*)$, respectively, are

$$u^+ = [1+\phi(E^+)]\ln c_o^+ \qquad \text{and} \qquad u^* = [1+\phi(E^*)]\ln c_o^*$$

So, if $c_o^+ \geq c_o^*$ then it is obvious that

$$[1+\phi(E^+)]\ln c_o^+ > [1+\phi(E^*)]\ln c_o^*$$

If $c_o^+ < c_o^*$ then $\exists E^{**} \in (E^*, E^+)$ such that

$$[1+\phi(E^{**})]c_o^+ = [1+\phi(E^*)]c_o^* \tag{54}$$

We know that the function $g(c) = \frac{\ln c}{c}$ is decreasing for $c > \mathrm{e}$ since $g'(c) = \frac{1-\ln c}{c^2} < 0\ \forall c > \mathrm{e}$ and $g(c)$ gets maximum at $c = \mathrm{e}$. So in case $c_o^* > c_o^+ \geq \mathrm{e}$, therefore, we have

$$\frac{\ln c_o^+}{c_o^+} > \frac{\ln c_o^*}{c_o^*} \tag{55}$$

From (54) and (55) we have $[1+\phi(E^{**})]\ln c_o^+ > [1+\phi(E^*)]\ln c_o^*$. Since $E^{**} < E^+$ then

$$[1+\phi(E^+)]\ln c_o^+ > [1+\phi(E^{**})]\ln c_o^+ \quad \Longrightarrow \quad [1+\phi(E^+)]\ln c_o^+ > [1+\phi(E^*)]\ln c_o^*$$

which contradicts the result that $(c_y^*, c_o^*, k_c^*, k_d^*, E^*)$ is the first-best steady state. Therefore $m \leq 1$ which implies that $k_d^* \leq A_d\alpha^{\frac{1}{1-\alpha}}$.

(ii) We prove $E^* \geq E_+^*$ by providing a negation. Suppose that $E^* < E_+^*$ then it is quite similar to the proof for statement (i) to obtain $[1+\phi(E^+)]\ln c_o^+ > [1+\phi(E^*)]\ln c_o^*$. This implies a contradiction that $(c_y^*, c_o^*, k_c^*, k_d^*, E^*)$ is not the first-best steady state. Therefore, $E^* \geq E_+^*$. Q.E.D.

7 Escaping the poverty-environment trap

This section studies a fiscal strategy that will enable an economy locked in a poverty-environment trap to escape stagnation. In this strategy, the government imposes a tax on the production of dirty intermediate inputs to improve environmental quality. By this way, the life expectancy of the agent is improved, thus encouraging savings in the long run. As will be shown later, when this tax rate is strong enough to improve environmental quality (and hence improve life expectancy) significantly, the interactions between environmental quality, life expectancy and capital accumulation enable the economy to escape the poverty-environment trap in the long run. Along with imposing a tax on the production of dirty intermediate inputs, the government can use this tax revenue in an efficient way by subsidizing the production of clean intermediate inputs.

We define that an economy falls into a poverty-environment trap when it converges to a competitive steady state that is characterized by low environmental quality, specifically $E < E_m$ as depicted in Figures 6b and 6d.[15] Let us consider the equilibrium of the economy under the tax (subsidy) imposed on the intermediate sectors. Let τ_{it} stand for the Pigovian tax rate (or subsidy, if negative) imposed on the production of intermediate input $i \in \{c, d\}$ in the period t. Under this tax, the monopolist maximizes her profits as follow

$$\pi_{it} = \max_{k_{it}} (1-\tau_i)\alpha A_i^{1-\alpha} k_{it}^{\alpha} - r_t k_{it}$$

So the rental rate of capital and the capital employed in the intermediate sector $i \in \{c, d\}$ are

[15] In this benchmark model, we highlight the strategy to help an economy move from stable steady state with low environmental quality to the other one with better environmental quality, then we define a poverty-environment trap is a stable steady state with $E < E_m$. In the later, we will redefine the poverty-environment trap is a stable steady state with $E < \hat{E}$.

$$r_t = (1-\tau_i)\alpha^2 A_i^{1-\alpha} k_{it}^{\alpha-1} \quad \text{and} \quad k_{it} = \left[\frac{(1-\tau_i)\alpha^2}{r_t}\right]^{\frac{1}{1-\alpha}} A_i$$

We have

$$k_t = k_{ct} + k_{dt} = \left(\frac{\alpha^2}{r_t}\right)^{\frac{1}{1-\alpha}} \left[(1-\tau_c)^{\frac{1}{1-\alpha}} A_c + (1-\tau_d)^{\frac{1}{1-\alpha}} A_d\right]$$

$$r_t = \alpha^2 \left[\frac{(1-\tau_d)^{\frac{1}{1-\alpha}} A_d + (1-\tau_c)^{\frac{1}{1-\alpha}} A_c}{k_t}\right]^{1-\alpha} \tag{56}$$

$$k_{it} = \frac{(1-\tau_i)^{\frac{1}{1-\alpha}} A_i}{(1-\tau_c)^{\frac{1}{1-\alpha}} A_c + (1-\tau_d)^{\frac{1}{1-\alpha}} A_d} k_t$$

The balanced budget constraint of the government in any period t requires it to hold

$$\tau_c \alpha A_c^{1-\alpha} k_{ct}^{\alpha} + \tau_d \alpha A_d^{1-\alpha} k_{dt}^{\alpha} = 0 \quad \Leftrightarrow \quad A_c(1-\tau_c)^{\frac{\alpha}{1-\alpha}}\tau_c + A_d(1-\tau_d)^{\frac{\alpha}{1-\alpha}}\tau_d = 0 \tag{57}$$

Under this policy, the monopoly profit of each intermediate sector $i \in \{c, d\}$ is

$$\pi_{it} = \alpha(1-\alpha)(1-\tau_i)^{\frac{1}{1-\alpha}} A_i \left[\frac{k_t}{(1-\tau_d)^{\frac{1}{1-\alpha}} A_d + (1-\tau_c)^{\frac{1}{1-\alpha}} A_c}\right]^{\alpha}$$

And the return to labor in the final good sector is

$$w_t = (1-\alpha)\frac{(1-\tau_d)^{\frac{\alpha}{1-\alpha}} A_d + (1-\tau_c)^{\frac{\alpha}{1-\alpha}} A_c}{\left[(1-\tau_d)^{\frac{1}{1-\alpha}} A_d + (1-\tau_c)^{\frac{1}{1-\alpha}} A_c\right]^{\alpha}} k_t^{\alpha}$$

Since from (57), $A_c(1-\tau_c)^{\frac{\alpha}{1-\alpha}}\tau_c + A_d(1-\tau_d)^{\frac{\alpha}{1-\alpha}}\tau_d = 0$, then we can rewrite w_t as

$$w_t = (1-\alpha)\left[(1-\tau_d)^{\frac{1}{1-\alpha}} A_d + (1-\tau_c)^{\frac{1}{1-\alpha}} A_c\right]^{1-\alpha} k_t^{\alpha}$$

The total income of the representative agent in the economy is now

$$I_t = (1-\alpha^2)\left[(1-\tau_d)^{\frac{1}{1-\alpha}} A_d + (1-\tau_c)^{\frac{1}{1-\alpha}} A_c\right]^{1-\alpha} k_t^{\alpha}$$

And the dynamics of the economy under this policy is now

$$k_{t+1} = \frac{\phi(E_t)(1-\alpha^2)}{1+\phi(E_t)}\left[(1-\tau_d)^{\frac{1}{1-\alpha}} A_d + (1-\tau_c)^{\frac{1}{1-\alpha}} A_c\right]^{1-\alpha} k_t^{\alpha}$$

$$E_{t+1} = E_t + \psi(E_t) - \frac{\xi(1-\tau_d)^{\frac{1}{1-\alpha}} A_d}{\left[(1-\tau_d)^{\frac{1}{1-\alpha}} A_d + (1-\tau_c)^{\frac{1}{1-\alpha}} A_c\right]^{\alpha}} \frac{\phi(E_t)(1-\alpha^2)}{1+\phi(E_t)} k_t^{\alpha}$$

Under this policy, the corresponding loci $KK' \equiv \left\{(k_t, E_t) \in \mathbb{R}_+^2 : k_{t+1} = k_t\right\}$ and $EE' \equiv \left\{(k_t, E_t) \in \mathbb{R}_+^2 : E_{t+1} = E_t\right\}$ are now characterized by

$$k_t = \left[\frac{\phi(E_t)(1-\alpha^2)}{1+\phi(E_t)}\right]^{\frac{1}{1-\alpha}} \left[(1-\tau_d)^{\frac{1}{1-\alpha}} A_d + (1-\tau_c)^{\frac{1}{1-\alpha}} A_c\right] \quad \text{or} \quad k_t = 0 \qquad (KK')$$

$$k_t = \left[\frac{\psi(E_t)[1+\phi(E_t)]}{\phi(E_t)(1-\alpha^2)\xi(1-\tau_d)^{\frac{1}{1-\alpha}} A_d} \right]^{\frac{1}{\alpha}} \left[(1-\tau_d)^{\frac{1}{1-\alpha}} A_d + (1-\tau_c)^{\frac{1}{1-\alpha}} A_c \right] \qquad (EE')$$

And the steady states under this policy are determined by

$$\xi A_d (1-\tau_d)^{\frac{1}{1-\alpha}} = \psi(E) \left[\frac{1+\phi(E)}{\phi(E)(1-\alpha^2)} \right]^{\frac{1}{1-\alpha}} \equiv G(E)$$

The strategy for such an economy to escape the poverty-environment trap is to tax on the production of dirty intermediate inputs in order to improve environmental quality exceeding E_m and converging to a high environmental quality. So, we fix the tax rate imposed on the production of dirty intermediate inputs so as to $\xi A_d(1-\tau_d)^{\frac{1}{1-\alpha}} = \hat{G} \in (0, G(E_m))$. Hence,

$$\tau_d = 1 - \left(\frac{\hat{G}}{\xi A_d} \right)^{1-\alpha} \in (0,1) \tag{58}$$

And $\tau_c < 0$ is uniquely determined by the following equation:

$$(1-\tau_c)^{\frac{\alpha}{1-\alpha}} \tau_c = -\frac{A_d}{A_c} \left(\frac{\hat{G}}{\xi A_d} \right)^{\alpha} \left[1 - \left(\frac{\hat{G}}{\xi A_d} \right)^{1-\alpha} \right] \tag{59}$$

Since $1-\tau_d = \left(\frac{\hat{G}}{\xi A_d}\right)^{1-\alpha} \leq \left(\frac{G(E_m)}{\xi A_d}\right)^{1-\alpha}$ then $\xi A_d(1-\tau_d)^{\frac{1}{1-\alpha}} \leq G(E_m)$ which implies that there will be one steady state $E > E_M$, i.e. the policy $(\tau_c, \tau_d)_t$ above guarantees the economy to escape the poverty-environment trap as depicted in the following figure.

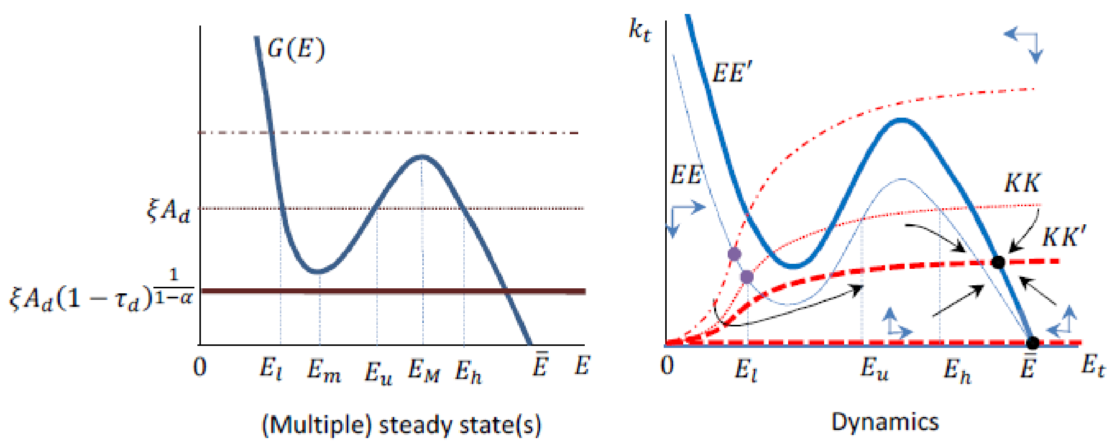

Fig 8. Escaping poverty-environment trap

We show that the new steady state to which the economy converges under the policy above $(\tau_c, \tau_d)_t$ can be locally stable. In effect, the Jacobian matrix evaluated around the new steady state of the new dynamic system under the policy $(\tau_c, \tau_d)_t$ is

$$J' = \begin{pmatrix} \alpha & \frac{\phi'(E)(1-\alpha^2)\left[(1-\tau_d)^{\frac{1}{1-\alpha}} A_d + (1-\tau_c)^{\frac{1}{1-\alpha}} A_c\right]\left[\frac{\phi(E)(1-\alpha^2)}{1+\phi(E)}\right]^{\frac{\alpha}{1-\alpha}}}{[1+\phi(E)]^2} \\ -\alpha \frac{\psi(E)\left[\frac{1+\phi(E)}{\phi(E)(1-\alpha^2)}\right]^{\frac{1}{1-\alpha}}}{(1-\tau_d)^{\frac{1}{1-\alpha}} A_d + (1-\tau_c)^{\frac{1}{1-\alpha}} A_c} & 1 + \psi'(E) - \frac{\psi(E)\phi'(E)}{\phi(E)[1+\phi(E)]} \end{pmatrix}$$

where $\det(J') = \alpha[1+\psi'(E)]$ and $Tr(J') = \alpha+1+\psi'(E) - \frac{\psi(E)\phi'(E)}{\phi(E)[1+\phi(E)]}$, which are exactly the same as the determinant and trace of the Jacobian matrix J as defined in Subsection 5.2. Hence, the stability/unstability type of the steady states in the new dynamic system and the former (without any tax) are exactly the same. Hence, we can find a proper policy $(\tau_c, \tau_d)_t$ such that the economy converges to a locally stable steady state. For example, we choose $\hat{G} \in (0, \min\{G(E_m), G(\hat{E})\})$ which guarantees that the economy converges to a locally stable steady state with $E > \hat{E}$. The stability of this steady state was discussed in Subsection 5.2.

Proposition 3: *In the overlapping generations economy with environmental externality as set up above, in the case $\|S\| = 2$ and $\xi A_d > G(E_m)$ and the economy is locked in a poverty-environment trap, then the following period-by-period balanced budget policy $(\tau_d, \tau_c)_t$ imposed on the production of the intermediate inputs,*

$$\tau_d = 1 - \left(\frac{\hat{G}}{\xi A_d}\right)^{1-\alpha} > 0 \quad \text{and} \quad (1-\tau_c)^{\frac{\alpha}{1-\alpha}}\tau_c = -\frac{A_d}{A_c}\left(\frac{\hat{G}}{\xi A_d}\right)^{\alpha}\left[1 - \left(\frac{\hat{G}}{\xi A_d}\right)^{1-\alpha}\right],$$

where $\tau_c < 0$ and $\hat{G} \in (0, G(E_m))$, can help the economy escape the poverty-environment trap.

Proof: The proof is presented above before Proposition 3 is stated.

8 Policy implementation of the first-best steady state

This section introduces fiscal policies in order to implement the first-best steady state. We focus on the economies that satisfy assumptions $(A4)$ and $(A5)$ which guarantee that environmental quality in the first-best steady state will be greater than the one at which the regeneration of the environment gets maximum, i.e. $E^* > \hat{E}$. We shall consider two cases. In the first case, we consider a fiscal policy applied to the economies converging at an interior state, which is closed to the first-best steady state, specifically environmental quality characterized by $E_t > \hat{E}$ for all $t \geq T'$ with some $T' \in \mathbb{N}$. In the second case, we consider the economies locked in a poverty-environment trap as mentioned in Section 7.

8.1 Policy implementation for economies converging to a high interior competitive steady state $E \in (\hat{E}, \bar{E})$

For this case, under competitive environment without any policy intervention, there always exists a period T' such that $E_t^c \in [\hat{E}, \bar{E}]$ and $k_{dt}^c < \frac{\psi(E)}{\xi}$ for all $t \geq T'$.[16] Suppose that from $T \geq T'$ onwards, we start to introduce a fiscal policy to implement the first-best steady state for that economy. In any period $t \geq T$, we introduce the Pigouvian tax (subsidy) τ_{it} which is imposed on the production of intermediate inputs $i \in \{c, d\}$. Under this tax, similar to the policy presented in Section 6, the allocation of capital for producing intermediate inputs $i \in \{c, d\}$ in period $t \geq T$ is

$$k_{it} = \frac{(1-\tau_{it})^{\frac{1}{1-\alpha}} A_i k_t}{(1-\tau_{dt})^{\frac{1}{1-\alpha}} A_d + (1-\tau_{ct})^{\frac{1}{1-\alpha}} A_c} \tag{60}$$

And the total income of the representative agent in period t is now

[16] We use the superscript "c" for state variables E_t and k_{dt} to mention these variables under the pure competitive environment.

$$I_t = (1-\alpha)\frac{(1-\tau_{dt})^{\frac{\alpha}{1-\alpha}}A_d + (1-\tau_{ct})^{\frac{\alpha}{1-\alpha}}A_c + \alpha\left[(1-\tau_{dt})^{\frac{1}{1-\alpha}}A_d + (1-\tau_{ct})^{\frac{1}{1-\alpha}}A_c\right]}{\left[(1-\tau_{dt})^{\frac{1}{1-\alpha}}A_d + (1-\tau_{ct})^{\frac{1}{1-\alpha}}A_c\right]^{\alpha}}k_t^{\alpha}$$

Along with the Pigouvian tax (subsidy) τ_{it} imposed on the production of intermediate inputs $i \in \{c, d\}$, in each period $t \geq T$ we also introduce a tax (subsidy) τ_t on consumptions, and another tax (subsidy) τ_{kt} on capital income. Under these taxes, the utility maximization problem of the agent born in period $t \geq T$ now is

$$\max_{c_{yt}, s_t, c_{ot+1}} \ln c_{yt} + \phi(E_t)\ln c_{ot+1}$$

$$\text{subject to}: \quad (1+\tau_t)c_{yt} + s_t = I_t \quad \text{and} \quad (1+\tau_{t+1})c_{ot+1} = \frac{r_{t+1}}{\phi(E_t)}s_t(1-\tau_{kt+1})$$

for given I_t, E_t, and r_{t+1}.

The agent's optimal choice under the tax policy

$$c_{yt} = \frac{I_t}{[1+\phi(E_t)](1+\tau_t)}; \quad s_t = \frac{\phi(E_t)}{1+\phi(E_t)}I_t; \quad c_{ot+1} = \frac{r_{t+1}(1-\tau_{kt+1})}{[1+\phi(E_t)](1+\tau_{t+1})}I_t$$

Under the policy $(\tau_{ct+1}, \tau_{dt+1}, \tau_{kt+1}, \tau_{t+1})_{t\geq T}$, the competitive equilibrium allocation in period T, $(c_{yT}, c_{oT+1}, k_{cT+1}, k_{dT+1}, E_T)$, is characterized by

$$c_{yT} = \frac{(1-\alpha^2)A^{1-\alpha}(k_{cT}+k_{dT})^{\alpha}}{1+\phi(E_T)}$$

$$c_{oT+1} = \frac{\alpha^2\left[(1-\tau_{dT+1})^{\frac{1}{1-\alpha}}A_d + (1-\tau_{cT+1})^{\frac{1}{1-\alpha}}A_c\right]^{1-\alpha}(k_{cT+1}+k_{dT+1})^{\alpha}}{\phi(E_T)}\frac{1-\tau_{kT+1}}{1+\tau_{T+1}}$$

$$k_{cT+1} = \frac{\phi(E_T)(1-\alpha^2)}{1+\phi(E_T)}\frac{(1-\tau_{cT+1})^{\frac{1}{1-\alpha}}A_c}{(1-\tau_{dT+1})^{\frac{1}{1-\alpha}}A_d + (1-\tau_{cT+1})^{\frac{1}{1-\alpha}}A_c}A^{1-\alpha}(k_{cT}+k_{dT})^{\alpha}$$

$$k_{dT+1} = \frac{\phi(E_T)(1-\alpha^2)}{1+\phi(E_T)}\frac{(1-\tau_{dT+1})^{\frac{1}{1-\alpha}}A_d}{(1-\tau_{dT+1})^{\frac{1}{1-\alpha}}A_d + (1-\tau_{cT+1})^{\frac{1}{1-\alpha}}A_c}A^{1-\alpha}(k_{cT}+k_{dT})^{\alpha}$$

$$E_T = E_{T-1} + \psi(E_{T-1}) - \xi k_{dT}$$

and the competitive equilibrium allocation of the economy from period $t \geq T+1$ onwards can be fully characterized by

$$c_{yt} = (1-\alpha)\frac{(1-\tau_{dt})^{\frac{\alpha}{1-\alpha}}A_d + (1-\tau_{ct})^{\frac{\alpha}{1-\alpha}}A_c + \alpha\left[(1-\tau_{dt})^{\frac{1}{1-\alpha}}A_d + (1-\tau_{ct})^{\frac{1}{1-\alpha}}A_c\right]}{\left[(1-\tau_{dt})^{\frac{1}{1-\alpha}}A_d + (1-\tau_{ct})^{\frac{1}{1-\alpha}}A_c\right]^{\alpha}[1+\phi(E_t)](1+\tau_t)}(k_{ct}+k_{dt})^{\alpha} \tag{61}$$

$$c_{ot+1} = \frac{\alpha^2\left[(1-\tau_{dt+1})^{\frac{1}{1-\alpha}}A_d + (1-\tau_{ct+1})^{\frac{1}{1-\alpha}}A_c\right]^{1-\alpha}(k_{ct+1}+k_{dt+1})^{\alpha}}{\phi(E_t)}\frac{1-\tau_{kt+1}}{1+\tau_{t+1}} \tag{62}$$

$$k_{ct+1} = \frac{\phi(E_t)(1-\alpha)}{1+\phi(E_t)} \frac{(1-\tau_{ct+1})^{\frac{1}{1-\alpha}} A_c \left[\frac{(1-\tau_{dt})^{\frac{\alpha}{1-\alpha}} A_d + (1-\tau_{ct})^{\frac{\alpha}{1-\alpha}} A_c}{(1-\tau_{dt})^{\frac{1}{1-\alpha}} A_d + (1-\tau_{ct})^{\frac{1}{1-\alpha}} A_c} + \alpha \right] (k_{ct}+k_{dt})^\alpha}{\left[(1-\tau_{dt+1})^{\frac{1}{1-\alpha}} A_d + (1-\tau_{ct+1})^{\frac{1}{1-\alpha}} A_c \right]^\alpha} \tag{63}$$

$$k_{dt+1} = \frac{\phi(E_t)(1-\alpha)}{1+\phi(E_t)} \frac{(1-\tau_{dt+1})^{\frac{1}{1-\alpha}} A_d \left[\frac{(1-\tau_{dt})^{\frac{\alpha}{1-\alpha}} A_d + (1-\tau_{ct})^{\frac{\alpha}{1-\alpha}} A_c}{(1-\tau_{dt})^{\frac{1}{1-\alpha}} A_d + (1-\tau_{ct})^{\frac{1}{1-\alpha}} A_c} + \alpha \right] (k_{ct}+k_{dt})^\alpha}{\left[(1-\tau_{dt+1})^{\frac{1}{1-\alpha}} A_d + (1-\tau_{ct+1})^{\frac{1}{1-\alpha}} A_c \right]^\alpha} \tag{64}$$

$$E_t = E_{t-1} + \psi(E_{t-1}) - \xi k_{dt} \tag{65}$$

for some given k_{cT}, k_{dT}, and E_{T-1}.

Proposition 4: *Under assumptions (A1), (A2), (A4), and (A5), in the overlapping generations economy with the environmental externality set up above, if the economy converges to an interior competitive steady state characterize by $E_h \in (\hat{E}, \bar{E})$, i.e. without any intervention, $\exists T'$ such that $\forall t \geq T'$, $E_t \in [\hat{E}, \bar{E}]$ and $k_{dt} < \frac{\psi(\hat{E})}{\xi}$ holds, then the first-best steady state can be attained by implementing the following period-by-period balanced-budget policy: Announcing in any period $t-1 \geq T'-1$ that the following tax rates on the production of the intermediate goods, capital income, and consumptions $(\tau_{ct}, \tau_{dt}, \tau_{kt}, \tau_t)$ will be applied in period t,*

$$\tau_{ct} = 1 - \frac{1}{1-\alpha} \frac{1+\phi(E^*)}{\phi(E^*)} + \frac{1}{\alpha} \frac{A_c + A_d m^{\frac{\alpha}{1-\alpha}}}{A_c + A_d m^{\frac{1}{1-\alpha}}} \equiv \tau_c^* \tag{66}$$

$$\tau_{dt} = 1 - \frac{m}{1-\alpha} \frac{1+\phi(E^*)}{\phi(E^*)} + \frac{m}{\alpha} \frac{A_c + A_d m^{\frac{\alpha}{1-\alpha}}}{A_c + A_d m^{\frac{1}{1-\alpha}}} \equiv \tau_d^* \tag{67}$$

$$\tau_{kt} = 1 - \frac{\phi(E_{t-1})(1-\alpha)}{[1+\phi(E_{t-1})]\alpha^2} \left[\frac{1}{1-\tau_c^*} \frac{A_c + A_d m^{\frac{\alpha}{1-\alpha}}}{A_c + A_d m^{\frac{1}{1-\alpha}}} + \alpha \right] \tag{68}$$

$$\tau_t = \frac{\frac{\phi(E_{t-1})}{1+\phi(E_{t-1})} + \frac{1}{1+\phi(E_t)}}{\frac{\alpha}{1-\alpha} \left(1 - \left[\frac{1}{1-\tau_c^*} \frac{A_c + A_d m^{\frac{\alpha}{1-\alpha}}}{A_c + A_d m^{\frac{1}{1-\alpha}}} + \alpha \right]^{-1} \right) + \frac{1}{1+\phi(E_t)}} - 1 \tag{69}$$

where $m = \frac{\psi'(E^*)}{\psi'(E^*) - \phi'(E^*)[\ln c_o^* - 1]\xi c_o^*}$.

***Proof*:** See Appendix A3.

The proof for Proposition 4 shows that whenever the economy has environmental quality $E_t \in [\hat{E}, \bar{E}]$ and capital that is allocated to produce dirty intermediate inputs $k_{dt} \in (0, \frac{\psi(\hat{E})}{\xi})$ then the policy introduced in Proposition 4 will help the economy converge to the first-best steady state. This suggests that, in order to implement the first-best steady state for economies converging to a low and stable steady state with $E < \frac{\psi(\hat{E})}{\xi}$, we need another policy that can help the economies move to the area satisfying $E_t \in [\hat{E}, \bar{E}]$ and $k_{dt} \in (0, \frac{\psi(\hat{E})}{\xi})$; therefore, we apply the policy introduced in Proposition 4 to help the economies converge to the first-best steady state. Such a strategy is presented and discussed in the next subsection.

8.2 Policy implementation for economies locked in a poverty-environment trap

Now, imagine that we want to implement the first-best steady state for an economy that is locked in a poverty-environment trap characterized by $E < E_m$ as defined in Section 7. The strategy in this case is more complex than the previous one and consists of two stages: (i) we introduce taxes $(\tau_c, \tau_d)_t$ similar to the ones in Section 7 so as to help the economy escape the poverty-environment trap; (ii) when the economy escapes the poverty-environment trap, we introduce a policy $(\tau_{ct}, \tau_{dt}, \tau_{kt}, \tau_t)_t$ similar to one in Subsection 8.1 to help the economy converge to the first-best steady state.

Proposition 5: *Under assumptions (A1), (A2), (A3), (A4), and (A5), in the overlapping generations economy with the environmental externality set up above, if the economy converges to an interior competitive steady state characterize by $E_l \in (0, E_m)$, i.e. the economy is locked in the poverty-environment trap, then the first-best steady state can be attained by implementing the following strategies:*

(i) *In any t, introduce the balanced-budget policy (τ_c, τ_d) (as introduced in Proposition 3)*

$$\tau_d = 1 - \left(\frac{\hat{G}}{\xi A_d}\right)^{1-\alpha} > 0 \quad \text{and} \quad (1-\tau_c)^{\frac{\alpha}{1-\alpha}}\tau_c = -\frac{A_d}{A_c}\left(\frac{\hat{G}}{\xi A_d}\right)^{\alpha}\left[1 - \left(\frac{\hat{G}}{\xi A_d}\right)^{1-\alpha}\right],$$

where $\tau_c < 0 and \hat{G} \in (0, \min\{G(E_m), G(\hat{E})\})$, will help the economy escape the poverty-environment trap;[17]

(ii) *Under the stage (i), when the economy has already escaped the poverty-environment trap and is in the period T the environmental quality satisfies $E_T > \hat{E}$, then from period $T+1$ onwards, we replace the balanced budget policy $(\tau_c, \tau_d)_{t \le T}$ above with the balanced budget policy $(\tau_{ct}, \tau_{dt}, \tau_{kt}, \tau_t)_{t \ge T+1}$ (as introduced in Proposition 4) which is announced in any period $t-1$:*

$$\tau_{ct} = 1 - \frac{1}{1-\alpha}\frac{1+\phi(E^*)}{\phi(E^*)} + \frac{1}{\alpha}\frac{A_c + A_d m^{\frac{\alpha}{1-\alpha}}}{A_c + A_d m^{\frac{1}{1-\alpha}}} \equiv \tau_c^*$$

$$\tau_{dt} = 1 - \frac{m}{1-\alpha}\frac{1+\phi(E^*)}{\phi(E^*)} + \frac{m}{\alpha}\frac{A_c + A_d m^{\frac{\alpha}{1-\alpha}}}{A_c + A_d m^{\frac{1}{1-\alpha}}} \equiv \tau_d^*$$

$$\tau_{kt} = 1 - \frac{\phi(E_{t-1})(1-\alpha)}{[1+\phi(E_{t-1})]\alpha^2}\left[\frac{1}{1-\tau_c^*}\frac{A_c + A_d m^{\frac{\alpha}{1-\alpha}}}{A_c + A_d m^{\frac{1}{1-\alpha}}} + \alpha\right]$$

$$\tau_t = \frac{\frac{\phi(E_{t-1})}{1+\phi(E_{t-1})} + \frac{1}{1+\phi(E_t)}}{\frac{\alpha}{1-\alpha}\left(1 - \left[\frac{1}{1-\tau_c^*}\frac{A_c + A_d m^{\frac{\alpha}{1-\alpha}}}{A_c + A_d m^{\frac{1}{1-\alpha}}} + \alpha\right]^{-1}\right) + \frac{1}{1+\phi(E_t)}} - 1$$

where $m = \frac{\psi'(E^*)}{\psi'(E^*) - \phi'(E^*)[\ln c_o^* - 1]\xi c_o^*}$.

***Proof*:** (i) Similar to the proof in Proposition 3, in an economy locked in a poverty-environment trap, the balanced budget policy $(\tau_c, \tau_d)_t$ guarantees that the economy will escape the trap and

[17] Note that the taxes set here differ from the ones in Proposition 3 is the the term $\hat{G} \in (0, \min\{G(E_m), G(\hat{E})\})$ in stead of $\hat{G} \in (0, G(E_m))$, which guarantees that the economy will not only escape the poverty-environment trap but will also converge to an interior steady state characterized by $E > \hat{E}$.

allow it to converge to a steady state characterized by $E > \hat{E}$. That is to say, there exists a period T such that for all $t \geq T$ and under the balanced budget policy $(\tau_c, \tau_d)_t$, it holds that $E_t > \hat{E}$ and $k_{dt} < \frac{\psi(\hat{E})}{\xi}$.

(ii) From period $t \geq T+1$ we reset the tax rates $(\tau_c, \tau_d)_{t \leq T}$ by $(\tau_c^*, \tau_d^*)_{t \geq T+1}$, and introduce taxes on capital income and consumptions $(\tau_{kt}, \tau_t)_{t \geq T+1}$. The government's budget under this policy $(\tau_c^*, \tau_d^*, \tau_{kt}, \tau_t)_{t \geq T+1}$ is always balanced and this policy guarantees that the economy will converge to the first-best steady state, as proven in Proposition 4. Q.E.D.

Proposition 5 provides the full fiscal strategy for implementing the social optimum for an economy that is locked in a poverty-environment trap. The economic reasonings behind Proposition 5 are rather intuitive. The first stage of the fiscal strategy helps the economy improve its environmental quality by taxing the production of dirty intermediate inputs, thus improving the life expectancy of the agents which, in turn, encourages the savings (capital accumulation). Note that, this stage focuses on reallocating capital in the production of intermediate inputs to enable environmental quality to exceed $\hat{E}$. Under this stage of taxation, there is a trade-off between the quantity of the final output and environmental quality in any period because the allocation rule of capital under this tax and subsidy policy no longer equalizes the marginal productivities of the intermediate inputs. Hence, in the short run (and maybe medium run), the final output per capita decreases. However, this stage of taxation improves the life expectancy of the agents through improving environmental quality. Longer life expectancy has a positive impact on capital accumulation, which fosters economic growth. Therefore, in the long run, the final output increases and the economy converges to a steady state with higher environmental quality that is greater than $\hat{E}$, and maybe higher final output per capita. In the second stage of the fiscal strategy, we introduce consumption and capital income taxes to decentralize the first-best steady state. The taxes on the production of intermediate inputs are reset at constant rates (τ_c^*, τ_d^*) in order to guarantee the rule of capital allocation in producing intermediate inputs to coincide with that of the social planner. The tax rates (τ_c^*, τ_d^*) affect the savings of the agents because they affect life expectancy through their impacts on environmental quality. In the meantime, the taxes on consumption and capital income guarantee the agent's optimal choices will converge to those of the social planner. In other word, the first-best steady state will be achieved. It is not surprising that the taxes on consumption and capital income depend on τ_c^* and τ_d^* because the policy $(\tau_c^*, \tau_d^*, \tau_{kt}, \tau_t)$ also guarantee that the government's budget will be balanced.

9 The case of $N > 3$ distinct interior competitive steady states

Let us revisit the equation (39) in determining the interior competitive steady states

$$\xi A_d = \psi(E) \left[\frac{1+\phi(E)}{\phi(E)(1-\alpha^2)} \right]^{\frac{1}{1-\alpha}} \equiv G(E) \tag{70}$$

and the first derivative of $G(E)$

$$G'(E) = \frac{\psi(E)}{1-\alpha} \left[\frac{1+\phi(E)}{\phi(E)(1-\alpha^2)} \right]^{\frac{1}{1-\alpha}} \left[(1-\alpha)\frac{\psi'(E)}{\psi(E)} - \frac{\phi'(E)}{[1+\phi(E)]\phi(E)} \right]$$

We relax assumption $(A3)$, which says that $\phi'''(E) = 0$ and $\frac{\partial^2(\psi'(E)/\psi(E))}{\partial E_t^2} < 0$. This assumption, indeed, lack the backgrounds for the third derivatives of $\phi(E)$ and $\psi(E)$. Recall that this assumption guarantees the strict concavity of $\frac{\psi'(E)}{\psi(E)}$ and the strict convexity $\frac{\phi'(E)}{[1+\phi(E)]\phi(E)}$, which make the cardinal of the set $S \equiv \left\{ E \in (0, \bar{E}) : \ \hat{\Theta}(E) = (1-\alpha)\frac{\psi'(E)}{\psi(E)} - \frac{\phi'(E)}{[1+\phi(E)]\phi(E)} = 0 \right\}$, as defined in

Section 5, is always less than or equal to two. Hence, the number of interior competitive steady states, i.e. the number of solutions to $G(E) = \xi A_d$, is maximum of three. So, if at least one of two properties $\phi'''(E) = 0$ and $\frac{\partial^2(\psi'(E)/\psi(E))}{\partial E_t^2} < 0$ does not hold, the cardinal of S may be greater two; therefore the number of interior competitive steady states may be greater than three, and a continuum of steady state may occur. We will study the properties of steady states, as well as a policy for an economy to escape a poverty-environment trap. We will also look at a policy that can implement the first-best steady state in such cases.

Lemma 7, which is an extension of the Lemma 4, states some important properties.

Lemma 7: *In the overlapping generations economy with the environmental externality set up above, under (A1), (A2), (A4), and (A5) the following properties hold:*

(i) $\sup(S) < \hat{E}$;

(ii) *The number of interior competitive steady states satisfying* $E \geq \hat{E}$ *is at most one;*

(iii) *In the case of that only distinct steady states prevails, then the number of interior competitive steady state is always less than or equal to* $\|S\| + 1$.

***Proof*:** Properties (i) and (ii) in Lemma 7 can be proven in the exact same way as properties (i) and (ii) in Lemma 4. We now only prove the property (iii). Suppose that the economy set up above has $N > 3$ distinct interior competitive steady states and assume that $\mathbb{N} \ni \|S\| = \tilde{N} > 2$. Each interior competitive steady state can be characterized by a pair $(E, k) \gg (0, 0)$. Since $(0, 0) \ll (E_n, k_n)_{n \in \{1,...,N\}} \in KK \cap EE$, where KK and EE are defined in (29) and (30) respectively; i.e.

$$k_t = \left[\frac{\phi(E_t)(1-\alpha^2)}{1+\phi(E_t)}\right]^{\frac{1}{1-\alpha}} A \equiv \Omega(E_t) \qquad (KK)$$

$$k_t = \left[\frac{\psi(E_t)\left[1+\phi(E_t)\right]}{\phi(E_t)(1-\alpha^2)\xi A_d}\right]^{\frac{1}{\alpha}} A \equiv \Phi(E_t) \qquad (EE)$$

where $\Omega(E_t)$ is monotonically increasing in $E_t \in (0, +\infty)$ as proven in Lemma 1. So, without the loss of generality, we can assume that

$$(k_1, E_1) \ll (k_2, E_2) \ll ... \ll (k_N, E_N)$$

And we can also assume that $\tilde{E}_1, \tilde{E}_2, ..., \tilde{E}_{\tilde{N}} \in S$ have the following order

$$0 < \tilde{E}_1 < \tilde{E}_2 < ... < \tilde{E}_{\tilde{N}} < \bar{E}$$

where $\tilde{N} = \|S\|$.

We prove that if $E_1 \leq \tilde{E}_1$ then $E_n > \tilde{E}_1 \; \forall n \in \{2, ..., N\}$, that is to say there is at most of one interior steady state satisfying $E_1 \leq \tilde{E}_1$. In effect, if $E_2 \leq \tilde{E}_1$ then $E_1 < E_2 \leq \tilde{E}_1$. We know that $G'(E) < 0 \; \forall E \in (0, \tilde{E}_1)$ since $\hat{\Theta}(E) = (1-\alpha)\frac{\psi'(E)}{\psi(E)} - \frac{\phi'(E)}{[1+\phi(E)]\phi(E)} < 0 \; \forall E \in (0, \tilde{E}_1)$. Hence, $G(E_1) > G(E_2)$ which contradicts the property that E_1 and E_2 are solutions to (70), i.e. $G(E_1) = G(E_2)$. Therefore, $E_2 > \tilde{E}_1$ and hence $E_n > \tilde{E}_1 \; \forall n \in \{2, ..., N\}$. Similarly, we prove that if $E_N \geq \tilde{E}_{\tilde{N}}$ then $E_n < \tilde{E}_{\tilde{N}} \; \forall n \in \{1, ..., N-1\}$. In effect, suppose that $E_{N-1} \geq \tilde{E}_{\tilde{N}}$ then $E_N > E_{N-1} \geq \tilde{E}_{\tilde{N}}$. We know that $G'(E) < 0 \; \forall E \in (\tilde{E}_{\tilde{N}}, \bar{E})$ and $G'(\tilde{E}_{\tilde{N}}) = 0$ because $\hat{\Theta}(E) = (1-\alpha)\frac{\psi'(E)}{\psi(E)} - \frac{\phi'(E)}{[1+\phi(E)]\phi(E)} < 0 \; \forall E \in (\tilde{E}_{\tilde{N}}, \bar{E})$ and $\hat{\Theta}(\tilde{E}_{\tilde{N}}) = 0$. Therefore, $G(E_N) < G(E_{N-1})$ which contradicts the property that E_N and E_{N-1} are solutions to (70). Therefore, $E_{N-1} < \tilde{E}_{\tilde{N}}$ and hence $E_n < \tilde{E}_{\tilde{N}} \; \forall n \in \{1, ..., N-1\}$. We continue to prove property (iii) by supposing a negation that $N \geq \tilde{N} + 2$. In this case, there exists $n \in \{1, ..., N\}$ and $\tilde{n} \in \{1, ..., \tilde{N}\}$ such that

$\tilde{E}_{\tilde{n}} \le E_n < E_{n+1} \le \tilde{E}_{\tilde{n}+1}$. However, the function $G(E)$ is monotonic in $E \in [\tilde{E}_{\tilde{n}}, \tilde{E}_{\tilde{n}+1}]$, hence $G(E_n) \neq G(E_{n+1})$ (since $E_n < E_{n+1}$) which contradicts to the property that E_n and E_{n+1} are solutions to (70), i.e. $G(E_n) = G(E_{n+1})$. Therefore, $N < \tilde{N}+2$, i.e. $N \le \tilde{N}+1 = \|S\|+1$. Q.E.D.

The properties of dynamics for the pair $(E_t, k_t)_t$ in the phase diagram, as stated in Lemma 3 still hold in this case, i.e.

$$\text{(i)} \quad k_{t+1} - k_t \begin{cases} > 0 & \text{if} \quad 0 < k_t < \Omega(E_t) \\ = 0 & \text{if} \quad k_t = \Omega(E_t) \\ < 0 & \text{if} \quad k_t > \Omega(E_t) \end{cases} \quad \text{and} \quad \text{(ii)} \quad E_{t+1} - E_t \begin{cases} > 0 & \text{if} \quad k_t < \Phi(E_t) \\ = 0 & \text{if} \quad k_t = \Phi(E_t) \\ < 0 & \text{if} \quad k_t > \Phi(E_t) \end{cases}$$

The determination of steady states and the dynamics of the economy can be depicted in the following figures and phase diagrams

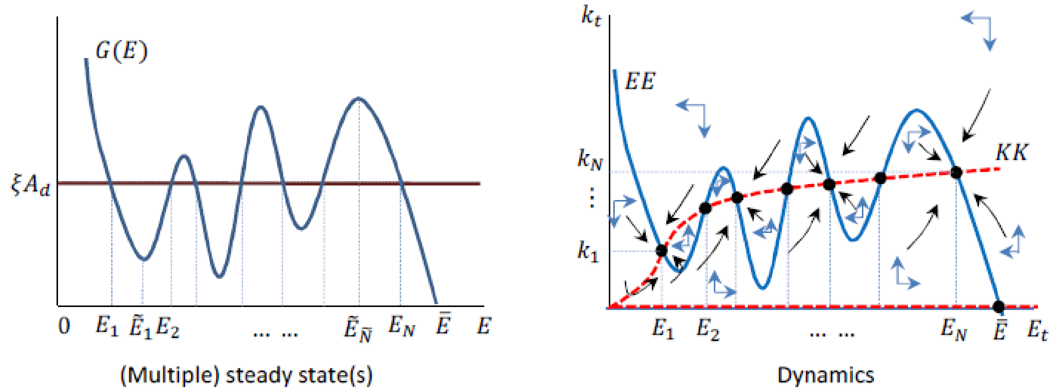

Fig 9a. Multiple distinct steady states and dynamics

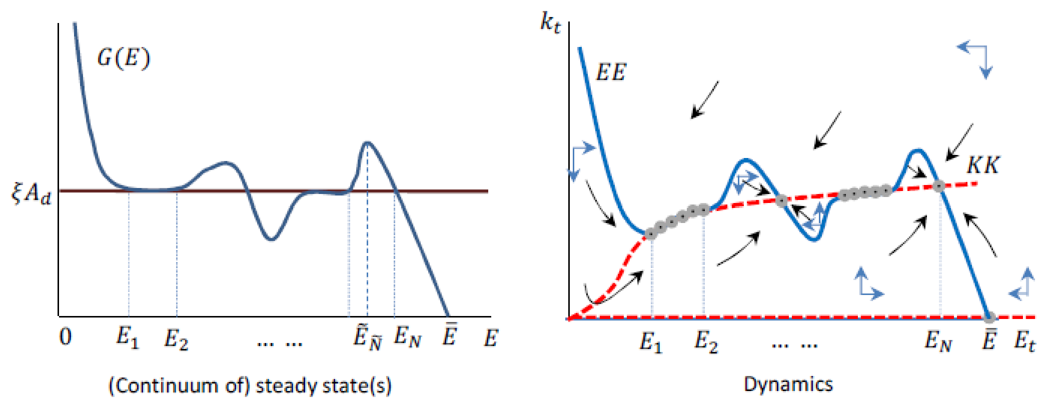

Fig 9b. Continuum of steady state and dynamics

The environmental regeneration and welfare properties of steady states in this case are similar to the benchmark case, which are stated in Proposition 2, particularly in the statement (iii), i.e. when the level of aggregate total factor productivity $A = A_c + A_d$ is high enough, then a competitive interior steady state associated with higher environmental quality provides the agent the higher stationary utility.

When the economy converges to a steady state with stationary environment quality $E > \hat{E}$ then we can apply the policy introduced in Proposition 4 to implement the first-best steady state whenever $E_t \in [\hat{E}, \bar{E}]$ and $k_{dt} \in (0, \frac{\psi(\hat{E})}{\xi})$.

Now it is interesting to study the policy in terms of its ability to free an economy from a poverty-environment. We redefine the economy that falls into a poverty-environment trap in Section 7 that when it converges to an interior competitive steady state characterized by $E < \tilde{E}_{\tilde{N}}$. Suppose that the economy converges to an interior competitive steady state with stationary environment quality E that satisfies $E \in (0, \hat{E})$. In other words, the economy is locked in a poverty-environment trap. The following proposition states the policy that can help such an economy escape the poverty-environment trap.

Proposition 6 (extension of Proposition 3 and Proposition 5): *In the case that $N > 3$ interior competitive steady states prevails and, if the economy converges to a stable steady state $E \in (0, \hat{E})$, i.e. the economy is locked in a poverty-environment trap, then the first-best steady state can be implemented by following two stages of taxation*

(i) ***Escaping poverty-environment trap:*** *Introduce a period-by-period balanced budget policy $(\tau_d, \tau_c)_t$ that is imposed on the production of intermediate inputs*

$$\tau_d = 1 - \left(\frac{\hat{G}}{\xi A_d}\right)^{1-\alpha} > 0 \quad \text{and} \quad (1-\tau_c)^{\frac{\alpha}{1-\alpha}}\tau_c = -\frac{A_d}{A_c}\left(\frac{\hat{G}}{\xi A_d}\right)^{\alpha}\left[1 - \left(\frac{\hat{G}}{\xi A_d}\right)^{1-\alpha}\right],$$

where $\tau_c < 0$ and $\hat{G} \in (0, \inf\{G(E') : E' \in (E, \hat{E}]\})$, will help the economy escape the poverty-environment trap, i.e. in some period T, $E_T \in (\hat{E}, \bar{E})$ and $k_{dT} < \frac{\psi(\hat{E})}{\xi}$.

(ii) ***Converging to the first-best steady state:*** *Under stage (i), when the economy has already escaped the poverty-environment trap and in period T when environmental quality satisfies $E_T > \hat{E}$, then from period $T+1$ onwards, we can replace the balanced budget policy $(\tau_c, \tau_d)_{t \leq T}$ above with the balanced budget policy $(\tau_{ct}, \tau_{dt}, \tau_{kt}, \tau_t)_{t \geq T+1}$ (as introduced in Proposition 4) which is announced in any period $t - 1 \geq T$:*

$$\tau_{ct} = 1 - \frac{1}{1-\alpha}\frac{1+\phi(E^*)}{\phi(E^*)} + \frac{1}{\alpha}\frac{A_c + A_d m^{\frac{\alpha}{1-\alpha}}}{A_c + A_d m^{\frac{1}{1-\alpha}}} \equiv \tau_c^*$$

$$\tau_{dt} = 1 - \frac{m}{1-\alpha}\frac{1+\phi(E^*)}{\phi(E^*)} + \frac{m}{\alpha}\frac{A_c + A_d m^{\frac{\alpha}{1-\alpha}}}{A_c + A_d m^{\frac{1}{1-\alpha}}} \equiv \tau_d^*$$

$$\tau_{kt} = 1 - \frac{\phi(E_{t-1})(1-\alpha)}{[1+\phi(E_{t-1})]\alpha^2}\left[\frac{1}{1-\tau_c^*}\frac{A_c + A_d m^{\frac{\alpha}{1-\alpha}}}{A_c + A_d m^{\frac{1}{1-\alpha}}} + \alpha\right]$$

$$\tau_t = \frac{\frac{\phi(E_{t-1})}{1+\phi(E_{t-1})} + \frac{1}{1+\phi(E_t)}}{\frac{\alpha}{1-\alpha}\left(1 - \left[\frac{1}{1-\tau_c^*}\frac{A_c + A_d m^{\frac{\alpha}{1-\alpha}}}{A_c + A_d m^{\frac{1}{1-\alpha}}} + \alpha\right]^{-1}\right) + \frac{1}{1+\phi(E_t)}} - 1$$

where $m = \frac{\psi'(E^*)}{\psi'(E^*) - \phi'(E^*)[\ln c_o^* - 1]\xi c_o^*}$.

Proof: We will provide proof for the case of there being only distinct interior steady states. For the case of continuum of steady state, the proof works analogously.

(i) Escaping the poverty-environment trap: First of all, we prove that under the tax policy $(\tau_d, \tau_c)_t$ introduced in stage (i), the economy, which is locked in an interior competitive steady state with $E \in (\tilde{E}_{\tilde{n}-1}, \tilde{E}_{\tilde{n}})$ where $\tilde{n} \in \{1, ..., \tilde{N}\}$,$\tilde{E}_0 = 0$, $\tilde{E}_{\tilde{n}} \in S$, can escape the poverty-environment trap and in some period T it will reach the state with $E_T \in (\hat{E}, \bar{E})$ and $k_{dT} < \frac{\psi(\hat{E})}{\xi}$. Indeed, suppose that, at the beginning of period $t = t_0$, the economy was already in such a stable steady state with $E \in (\tilde{E}_{\tilde{n}-1}, \tilde{E}_{\tilde{n}})$ when the policy (τ_d, τ_c) started to be applied. Under this policy, the capital allocated for the production of dirty intermediate inputs in period t_0 is

$$k_{dt_0} = \frac{(1-\tau_d)^{\frac{1}{1-\alpha}} A^\alpha}{\left[(1-\tau_d)^{\frac{1}{1-\alpha}} A_d + (1-\tau_c)^{\frac{1}{1-\alpha}} A_c\right]^\alpha} \frac{\phi(E_{t_0-1})(1-\alpha^2)}{1+\phi(E_{t_0-1})} A_d^{1-\alpha} k_{dt_0-1}^\alpha$$

$$= \frac{(1-\tau_d) A^\alpha}{\left[A_d + \left(\frac{1-\tau_c}{1-\tau_d}\right)^{\frac{1}{1-\alpha}} A_c\right]^\alpha} \frac{\phi(E)(1-\alpha^2)}{1+\phi(E)} A_d^{1-\alpha} k_{dt_0-1}^\alpha < \frac{\psi(E)}{\xi}$$

since $(1-\tau_d)A^\alpha / \left[A_d + \left(\frac{1-\tau_c}{1-\tau_d}\right)^{\frac{1}{1-\alpha}} A_c\right]^\alpha < 1$. Note that at a steady state $k_{dt_0-1} = k_d$ and $\frac{\phi(E)(1-\alpha^2)}{1+\phi(E)} A_d^{1-\alpha} k_d^\alpha = \frac{\psi(E)}{\xi}$.

Hence, in period t_0 we have $E_{t_0} > E_{t_0-1}$. If $E_{t_0} = E_{t_0-1} + \psi(E_{t_0-1}) - \xi k_{dt_0} \leq \hat{E}$; therefore $k_{dt_0} < \frac{\psi(E_{t_0})}{\xi}$. Under the policy (τ_d, τ_c) above, the capital allocated for the production of dirty intermediate inputs in period $t_0 + 1$ is

$$k_{dt_0+1} = \frac{\phi(E_{t_0})(1-\alpha^2)}{1+\phi(E_{t_0})} (1-\tau_d) A_d^{1-\alpha} k_{dt_0}^\alpha = \frac{\phi(E_{t_0})(1-\alpha^2)}{1+\phi(E_{t_0})} \left(\frac{\hat{G}}{\xi}\right)^{1-\alpha} k_{dt_0}^\alpha$$

Since $\hat{G} \in (0, \inf\{G(E') : E' \in (E, \hat{E}]\})$, then $\hat{G} < G(E_{t_0})$. We have above $k_{dt_0} < \frac{\psi(E_{t_0})}{\xi}$, therefore,

$$k_{dt_0+1} < \frac{\phi(E_{t_0})(1-\alpha^2)}{1+\phi(E_{t_0})} \left[\frac{\psi(E_{t_0})}{\xi}\right]^{1-\alpha} \frac{1+\phi(E_{t_0})}{\phi(E_{t_0})(1-\alpha^2)} \left[\frac{\psi(E_{t_0})}{\xi}\right]^\alpha = \frac{\psi(E_{t_0})}{\xi}$$

So by induction, in any period $t \geq t_0$, if $E_t \leq \hat{E}$ then $k_{dt+1} < \frac{\psi(E_t)}{\xi}$. That is to say, under the tax policy $(\tau_d, \tau_c)_t$ above, there exists a period T such that $E_T > \hat{E}$ and $k_{dT} < \frac{\psi(\hat{E})}{\xi}$.

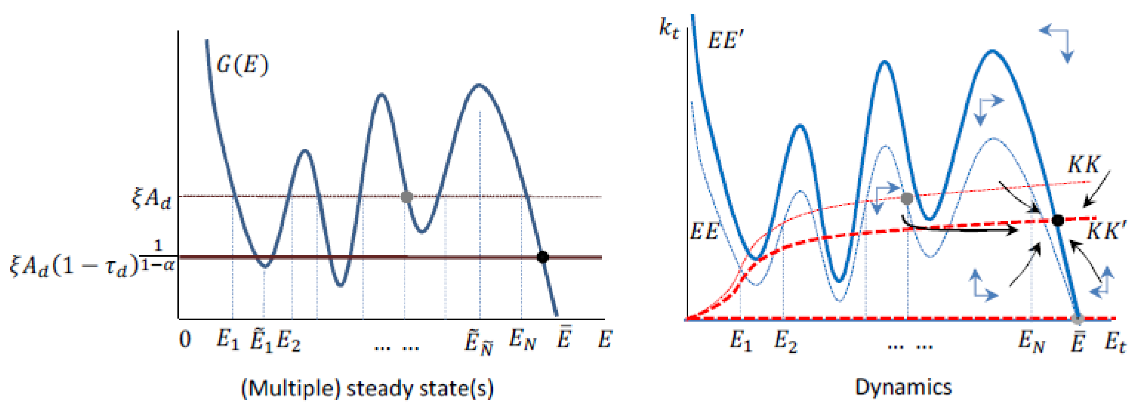

Fig 10a. Escaping poverty-environment trap in the case of multiple distinct steady states

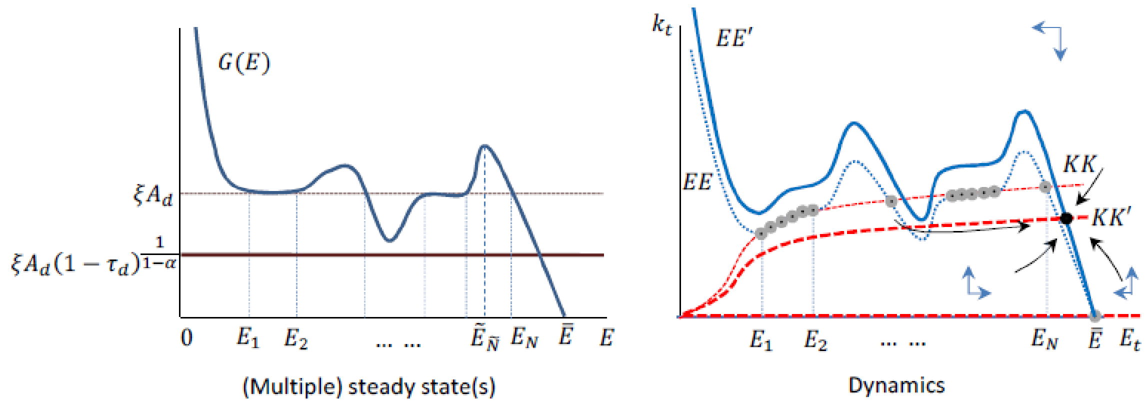

Fig 10b. Escaping poverty-environment trap in the case of continuum steady state

(ii) Converging to the first best steady state: From period $T+1$ onwards, the policy $(\tau_c^*, \tau_d^*, \tau_{kt}, \tau_t)_{t\geq T+1}$ will be implemented. The proof for the convergence to the first-best steady state of the economy under this policy is similar to the proof provided for Proposition 4. Q.E.D.

The economic reasonings for the fiscal strategy in this case are similar to those discussed after Proposition 5 in Subsection 8.2.

10 Conclusion

In this paper, we have studied the interactions between the environment, life expectancy, and capital accumulation in a dynamic general equilibrium model. The feedback loop of these factors is rather intuitive. The impact of environmental quality on the life expectancy of the agents makes them determine their optimal savings for their consumptions when old. Their savings, in terms of physical capital, degrades the environment through producing dirty intermediate inputs in the next period. Our theoretical results show the possibility for existing convergence clubs in terms of the environment and life expectancy, which matches across countries, as mentioned in Stylized fact 2. The mechanism for an economy to be locked in a poverty-environment trap (i.e. a low steady state) is initial low environmental quality lowers the agent's life expectancy and discourages them from savings, which ultimately inhibits capital accumulation, because agents tend to spend more when young. Although the stock of capital may be low, which leads to lower emissions, environmental quality is low because the environmental regeneration at the low environmental quality state can be completely offset by the emissions. A similar mechanism with opposite directions is applied to explain the counterparts that converge to high steady states.

Our paper proposes a tax and subsidy strategy to be imposed on the production of intermediate inputs to help economies that are locked in the poverty-environment trap to reallocate capital towards reducing the production of dirty intermediate inputs in order to improve environmental quality in both short-run and long-run. A better environment enhances life expectancy and fosters capital accumulation. Hence, under such a suitable period-by-period strategy, the interplay between the environment and capital accumulation through the life expectancy channel will help the economy escape the poverty-environment trap. In addition to guaranteeing that the economy will escape the poverty-environment trap, we propose another period-by-period balanced-budget fiscal strategy, which includes taxes (subsidies) on the production of intermediate inputs, consumptions and capital income, in order to decentralize the social optimum. The social optimum differs from the competitive equilibrium steady state because of imperfect altruism between generations in the competitive economy, and also because the social planner can internalize the impact of environmental externality on life expectancy, whereas individual agents in the economy cannot.

Our paper leaves some room for further research for both theory and empirics. (i) Of course, there are many other factors besides the environment that affect life expectancy, such as public healthcare, the improvement of medical technologies, the effect of income on health profile of people, etc. Incorporating these issues into our overlapping generations framework is a challenge, but it promises interesting results, as well as implications for environmental policy and other policies that affect the health profile of individuals. (ii) Our paper employs an exogenous growth model with two intermediate sectors and one final good. One could extend the model by including the possibilities of *quality-improving* (or *vertical innovations*) for the intermediate inputs, as well as for pollution abatement technology. This way of modeling may bear interesting strategies for allocating resources towards *research and development* and pollution abatement. (iii) Incorporating a demographic concern into this framework, i.e. allowing for endogenous fertility and relating population to the environment is also an ambitious but interesting extension. (iv) Introducing fixed factor(s) of production, such as "*land*", whose productivity may be affected by environmental quality in an overlapping generations model may provides us a more comprehensive fiscal strategy for sustainable economic growth.

Appendix

A1. Solving the problem of the social planner

The Lagrangian of the problem is

$$\mathcal{L} = \sum_{t=0}^{+\infty} \frac{\ln c_{yt} + \phi(E_t) \ln c_{ot+1}}{(1+R)^t} + \sum_{t=0}^{+\infty} \frac{\lambda_{2t}}{(1+R)^t} \left[E_t - E_{t-1} - \psi(E_{t-1}) + \xi k_{dt}\right]$$
$$+ \sum_{t=0}^{+\infty} \frac{\lambda_{1t}}{(1+R)^t} \left[A_c^{1-\alpha} k_{ct}^{\alpha} + A_d^{1-\alpha} k_{dt}^{\alpha} - c_{yt} - \phi(E_{t-1}) c_{ot} - k_{ct+1} - k_{dt+1}\right]$$

The FOCs are

$$\frac{\partial \mathcal{L}}{\partial c_{yt}} = \frac{1}{(1+R)^t} \frac{1}{c_{yt}} - \frac{\lambda_{1t}}{(1+R)^t} = 0$$

$$\frac{\partial \mathcal{L}}{\partial c_{ot}} = \frac{1}{(1+R)^{t-1}} \frac{\phi(E_{t-1})}{c_{ot}} - \frac{\lambda_{1t}}{(1+R)^t} \phi(E_{t-1}) = 0$$

$$\frac{\partial \mathcal{L}}{\partial k_{ct}} = \frac{\lambda_{1t}}{(1+R)^t} \alpha A_c^{1-\alpha} k_{ct}^{\alpha-1} - \frac{\lambda_{1t-1}}{(1+R)^{t-1}} = 0$$

$$\frac{\partial \mathcal{L}}{\partial k_{dt}} = \frac{\lambda_{1t}}{(1+R)^t} \alpha A_d^{1-\alpha} k_{dt}^{\alpha-1} - \frac{\lambda_{1t-1}}{(1+R)^{t-1}} + \frac{\lambda_{2t}}{(1+R)^t} \xi = 0$$

$$\frac{\partial \mathcal{L}}{\partial E_t} = \frac{\phi'(E_t) \ln c_{ot+1}}{(1+R)^t} - \frac{\lambda_{1t+1}}{(1+R)^{t+1}} \phi'(E_t) c_{ot+1} + \frac{\lambda_{2t}}{(1+R)^t} - \frac{\lambda_{2t+1}}{(1+R)^{t+1}} \left[1 + \psi'(E_t)\right] = 0$$

$$A_c^{1-\alpha} k_{ct}^{\alpha} + A_d^{1-\alpha} k_{dt}^{\alpha} = c_{yt} + \phi(E_{t-1}) c_{ot} + k_{ct+1} + k_{dt+1}$$

$$E_t = E_{t-1} + \psi(E_{t-1}) - \xi k_{dt}$$

i.e., at the steady state $(\bar{c}_y^*, \bar{c}_o^*, \bar{k}_c^*, \bar{k}_d^*, \bar{E}^*)$

$$\frac{1}{\bar{c}_y^*} = \lambda_1$$

$$\frac{1}{\bar{c}_o^*} = \frac{\lambda_1}{1+R}$$

$$\frac{\alpha A_c^{1-\alpha} \bar{k}_c^{*\alpha-1}}{1+R} = 1$$

$$\frac{\lambda_1}{1+R} \alpha A_d^{1-\alpha} \bar{k}_d^{*\alpha-1} - \lambda_1 + \frac{\lambda_2}{1+R} \xi = 0$$

$$\phi'(\bar{E}^*) \left(\ln \bar{c}_o^* - 1\right) + \lambda_2 - \frac{\lambda_2}{1+R} \left[1 + \psi'(\bar{E}^*)\right] = 0$$

$$A_c^{1-\alpha} \bar{k}_c^{*\alpha} + A_d^{1-\alpha} \bar{k}_d^{*\alpha} = \bar{c}_y^* + \phi(\bar{E}^*) \bar{c}_o^* + \bar{k}_c^* + \bar{k}_d^*$$

$$\bar{E}^* = \bar{E}^* + \psi(\bar{E}^*) - \xi \bar{k}_d^*$$

i.e., that is to say

$$\frac{\bar{c}_o^*}{\bar{c}_y^*} = 1 + R$$

$$\bar{k}_c^* = \left(\frac{\alpha}{1+R}\right)^{\frac{1}{1-\alpha}} A_c$$

$$\bar{k}_d^* = \left[\frac{\alpha \left[\psi'(\bar{E}^*) - R\right]}{(1+R) \left[\psi'(\bar{E}^*) - R\right] - \phi'(\bar{E}^*) \left(\ln \bar{c}_o^* - 1\right) \xi \bar{c}_o^*}\right]^{\frac{1}{1-\alpha}} A_d$$

$$A_c^{1-\alpha} \bar{k}_c^{*\alpha} + A_d^{1-\alpha} \bar{k}_d^{*\alpha} = \bar{c}_y^* + \phi(\bar{E}^*) \bar{c}_o^* + \bar{k}_c^* + \bar{k}_d^*$$

$$\psi(\bar{E}^*) = \xi \bar{k}_d^*$$

A2. Proof of Lemma 5:

(i) It is obvious that if $\psi'(E) \geq \frac{1}{\alpha} - 1$, i.e. $\det(J) = \alpha \left[1 + \psi'(E)\right] \geq 1$, then the corresponding steady state is unstable because there is at least one eigenvalue with an absolute value greater than 1 (if it is real), or with a modulus greater than 1 (if it is complex), or there are repeated eigenvalues with absolute values equal to 1.

(ii) Next, we consider the steady states with environmental qualities that satisfy $\psi'(E) < \frac{1}{\alpha} - 1$, i.e. $\det(J) \in (0, 1)$.

(ii.a) If $\hat{\Theta}(E) > 0$ then $Tr(J) > \det(J) + 1 > 1$. Since $\det(J) \in (0, 1)$ then $Tr(J)^2 > 4 \det(J)$, i.e. the Jacobian matrix has two distinct real eigenvalues $\lambda_2 > \lambda_1 > 0$. We prove in this case that $\lambda_2 > 1 > \lambda_1 > 0$. In effect, suppose that $\lambda_2 \leq 1$. If $\lambda_2 = 1$ then $Tr(J) = 1 + \lambda_1 = \det(J) + 1$ which contradicts to the inequality $Tr(J) > \det(J) + 1$. If $0 < \lambda_2 < 1$, we set $\lambda_2 = 1 - \varepsilon$ where $\varepsilon \in (0, 1)$, then

$$Tr(J) > \det(J) + 1 \quad \Leftrightarrow \quad 1 - \varepsilon + \lambda_1 > (1-\varepsilon)\lambda_1 + 1 \quad \Leftrightarrow \quad \lambda_1 > 1 \quad \Rightarrow \quad \lambda_1 > \lambda_2$$

which contradicts to the assumption $\lambda_1 < \lambda_2$. Therefore, $\lambda_2 > 1$. It is obvious that $0 < \lambda_1 < 1$ because $\lambda_1\lambda_2 = \det(J) \in (0,1)$. That is to say, the steady state in this case is a saddle point.

(ii.b) If $\hat{\Theta}(E) < 0$ then $Tr(J) < \det(J) + 1$. It is straightforward that if $Tr(J) \geq 0$ or $Tr(J)^2 \leq 4\det(J)$, i.e. $Tr(J) \geq -2\sqrt{\alpha\,[1+\psi'(E)]}$, then the Jacobian matrix has two eigenvalues with absolute values that are strictly less than 1 (if they are real), or with moduli that are strictly less than 1 (if they are complex). So the steady state is locally stable. We next consider the case $Tr(J) < -2\sqrt{\alpha\,[1+\psi'(E)]}$. In this case, the Jacobian matrix has 2 negative distinct eigenvalues. Therefore, it is sufficient for the steady state to be locally stable in this case that

$$-1 < \frac{Tr(J) - \sqrt{Tr(J)^2 - 4\det(J)}}{2}$$

That is, given $Tr(J) < -2\sqrt{\alpha\,[1+\psi'(E)]}$, the last inequality is equivalent to

$$Tr(J)^2 + 4Tr(J) + 4 > Tr(J)^2 - 4\det(J) \quad \Leftrightarrow \quad Tr(J) + 1 > -\det(J)$$

$$\Leftrightarrow\; 2\alpha\,[1+\psi'(E)] + 2 > \frac{\psi(E)\phi'(E)}{\phi(E)[1+\phi(E)]} - (1-\alpha)\psi'(E) \;\Leftrightarrow\; 1+\alpha > \frac{\psi(E)}{2+\psi'(E)}\frac{\phi'(E)}{\phi(E)[1+\phi(E)]}$$

Now, we prove that if $Tr(J) \geq -2\sqrt{\alpha\,[1+\psi'(E)]}$ then the last inequality must hold. In effect, we rewrite $Tr(J) \geq -2\sqrt{\alpha\,[1+\psi'(E)]}$ as

$$\alpha + 1 + \psi'(E) - \frac{\psi(E)\phi'(E)}{\phi(E)[1+\phi(E)]} \geq -2\sqrt{\alpha\,[1+\psi'(E)]}$$

$$\Rightarrow \quad (1+\alpha)\,[2+\psi'(E)] > \frac{\psi(E)\phi'(E)}{\phi(E)[1+\phi(E)]} \qquad \left(\text{since } 2\sqrt{\alpha\,[1+\psi'(E)]} < 1 + \alpha\,[1+\psi'(E)]\right)$$

$$\text{i.e.} \quad 1+\alpha > \frac{\psi(E)}{2+\psi'(E)}\frac{\phi'(E)}{\phi(E)[1+\phi(E)]}.$$

(iii) In the case of $E \in (\hat{E}, \bar{E})$, we have $\psi'(E) < 0$ and $\hat{\Theta}(E) < 0$,

$$\det(J) = \alpha\,[1+\psi'(E)] \in (0,\alpha) \qquad \text{and} \qquad Tr(J) = \det(J) + 1 + \hat{\Theta}(E)\psi(E) < \alpha\,[1+\psi'(E)] + 1$$

Indeed, there are two exclusive subcases for $Tr(J)$. They are, $-[\alpha + 1 + \psi'(E)] \leq Tr(J) < \alpha\,[1+\psi'(E)] + 1$ or $Tr(J) < -[\alpha + 1 + \psi'(E)]$.

It is straightforward that if $-[\alpha + 1 + \psi'(E)] \leq Tr(J) < \alpha\,[1+\psi'(E)] + 1$, then the Jacobian matrix has two eigenvalues with absolute values that are strictly less than 1 (if they are real), or with moduli that are strictly less than 1 (if they are complex); So the steady state is locally stable. Note that, if this subcase happens, the steady state is locally stable even the condition $1+\alpha \geq \frac{\psi(\hat{E})\phi'(\hat{E})}{\phi(\hat{E})[1+\phi(\hat{E})]}$ stated in (iii) does not hold.

Next, we consider the subcase $Tr(J) < -[\alpha + 1 + \psi'(E)]$. In this case, the Jacobian matrix has two negative distinct eigenvalues. In order for the steady state to be locally stable, it is sufficient that

$$-1 < \frac{Tr(J) - \sqrt{Tr(J)^2 - 4\det(J)}}{2}$$

That is, given $Tr(J) < -[\alpha + 1 + \psi'(E)]$, it is quite similar to the proof in (ii.b) we obtain

$$Tr(J)^2 + 4Tr(J) + 4 > Tr(J)^2 - 4\det(J) \quad \Leftrightarrow \quad (1+\alpha)\left[2 + \psi'(E)\right] > \frac{\psi(E)\phi'(E)}{\phi(E)[1+\phi(E)]} \tag{71}$$

Since $\forall E \in (\hat{E}, \bar{E})$ we have $2+\psi'(E) > 1$ and $\frac{\psi(E)\phi'(E)}{\phi(E)[1+\phi(E)]} < \frac{\psi(\hat{E})\phi'(\hat{E})}{\phi(\hat{E})[1+\phi(\hat{E})]}$, hence $1+\alpha \geq \frac{\psi(\hat{E})\phi'(\hat{E})}{\phi(\hat{E})[1+\phi(\hat{E})]}$ guarantees that (71) holds, i.e. the steady state is locally stable.

So $1+\alpha \geq \frac{\psi(\hat{E})\phi'(\hat{E})}{\phi(\hat{E})[1+\phi(\hat{E})]}$ is a *sufficient* condition for the steady state to be locally stable in this case. Q.E.D.

A3. Proof of Proposition 4:

We will prove that, (i) period by period, the tax policy $(\tau_c^*, \tau_d^*, \tau_{kt}, \tau_t)_t$ as described in Proposition 4 guarantees that the government's budget will be balanced; and (ii) that it will help an economy, *which converges to a competitive steady state characterized by* $E^c \in (\hat{E}, \bar{E})$, to converge to the first-best steady state and that the policy $(\tau_c^*, \tau_d^*, \tau_{kt}, \tau_t)_t$ will converge to $(\tau_c^*, \tau_d^*, \tau_k^*, \tau^*)$. Indeed, the net tax revenue for the government in any period $t \geq T$ is

$$B_t = \tau_c^* \alpha A_c^{1-\alpha} k_{ct}^{\alpha} + \tau_d^* \alpha A_d^{1-\alpha} k_{dt}^{\alpha} + \tau_{kt} r_t (k_{ct} + k_{dt}) + \tau_t [c_{yt} + \phi(E_{t-1}) c_{ot}] \tag{72}$$

where, similar to (56), the rental rate of capital is

$$r_t = \alpha^2 \left[\frac{(1-\tau_d^*)^{\frac{1}{1-\alpha}} A_d + (1-\tau_c^*)^{\frac{1}{1-\alpha}} A_c}{k_{ct} + k_{dt}} \right]^{1-\alpha} \tag{73}$$

By substituting $(\tau_c^*, \tau_d^*, \tau_{kt}, \tau_t)$ as defined in (66), (67), (68), (69), and $(c_{yt}^{\tau}, c_{ot}^{\tau}, k_{ct}^{\tau}, k_{dt}^{\tau})$ as defined in (61), (62), (60), and r_t as defined in (73) into (72) with a simple transformation, we obtain

$$\begin{aligned}
\frac{B_t}{\left[(1-\tau_d^*)^{\frac{1}{1-\alpha}} A_d + (1-\tau_c^*)^{\frac{1}{1-\alpha}} A_c\right]^{1-\alpha} (k_t^{\tau})^{\alpha}} &= \alpha \left[\frac{(1-\tau_d^*)^{\frac{\alpha}{1-\alpha}} A_d + (1-\tau_c^*)^{\frac{\alpha}{1-\alpha}} A_c}{(1-\tau_d^*)^{\frac{1}{1-\alpha}} A_d + (1-\tau_c^*)^{\frac{1}{1-\alpha}} A_c} - 1 \right] \\
&+ \tau_{kt}\alpha^2 + \frac{1-\alpha}{1+\phi(E_t)} \left(\frac{(1-\tau_d^*)^{\frac{\alpha}{1-\alpha}} A_d + (1-\tau_c^*)^{\frac{\alpha}{1-\alpha}} A_c}{(1-\tau_d^*)^{\frac{1}{1-\alpha}} A_d + (1-\tau_c^*)^{\frac{1}{1-\alpha}} A_c} + \alpha \right) + \alpha^2 (1-\tau_{kt}) \\
&- \frac{1}{1+\tau_t} \left[\frac{1-\alpha}{1+\phi(E_t)} \left(\frac{(1-\tau_d^*)^{\frac{\alpha}{1-\alpha}} A_d + (1-\tau_c^*)^{\frac{\alpha}{1-\alpha}} A_c}{(1-\tau_d^*)^{\frac{1}{1-\alpha}} A_d + (1-\tau_c^*)^{\frac{1}{1-\alpha}} A_c} + \alpha \right) + \alpha^2 (1-\tau_{kt}) \right] \\
&= \alpha(M-1) + \frac{1-\alpha}{1+\phi(E_t)} M - (1-\alpha) \left[\frac{\alpha\,(M-1)}{1-\alpha} + \frac{M}{1+\phi(E_t)} \right] = 0
\end{aligned}$$

where $M = \frac{(1-\tau_d^*)^{\frac{\alpha}{1-\alpha}} A_d + (1-\tau_c^*)^{\frac{\alpha}{1-\alpha}} A_c}{(1-\tau_d^*)^{\frac{1}{1-\alpha}} A_d + (1-\tau_c^*)^{\frac{1}{1-\alpha}} A_c} + \alpha = \frac{1}{1-\tau_c^*} \frac{A_c + A_d m^{\frac{\alpha}{1-\alpha}}}{A_c + A_d m^{\frac{1}{1-\alpha}}} + \alpha$. Hence, $B_t = 0$ which implies that the government budget is always balanced.

Moreover, the feasibility of resource allocation $(c_{yt}, c_{ot}, k_{ct+1}, k_{dt+1}, E_t)$ under the tax policy $(\tau_c^*, \tau_d^*, \tau_{kt}, \tau_t)_{t \geq T}$ always holds, in effect for given E_{t-1}, and k_{ct}, k_{dt} with allocation rule

$$\frac{k_{ct}}{(1-\tau_c^*)^{\frac{1}{1-\alpha}}A_c} = \frac{k_{dt}}{(1-\tau_d^*)^{\frac{1}{1-\alpha}}A_d} = \frac{k_t}{(1-\tau_c^*)^{\frac{1}{1-\alpha}}A_c + (1-\tau_d^*)^{\frac{1}{1-\alpha}}A_d}$$

From (61), (62), (63), and (64), we have

$$c_{yt} + \phi(E_{t-1})c_{ot} + k_{ct+1} + k_{dt+1} =$$

$$(1-\alpha)\frac{(1-\tau_d^*)^{\frac{\alpha}{1-\alpha}}A_d + (1-\tau_c^*)^{\frac{\alpha}{1-\alpha}}A_c + \alpha\left[(1-\tau_d^*)^{\frac{1}{1-\alpha}}A_d + (1-\tau_c^*)^{\frac{1}{1-\alpha}}A_c\right]}{\left[(1-\tau_d^*)^{\frac{1}{1-\alpha}}A_d + (1-\tau_c^*)^{\frac{1}{1-\alpha}}A_c\right]^\alpha [1+\phi(E_t)](1+\tau_t)} k_t^\alpha$$
$$+\left[\alpha^2\frac{1-\tau_{kt}}{1+\tau_t} + \frac{\phi(E_t)(1-\alpha)}{1+\phi(E_t)}M\right]\left[(1-\tau_d^*)^{\frac{1}{1-\alpha}}A_d + (1-\tau_c^*)^{\frac{1}{1-\alpha}}A_c\right]^{1-\alpha} k_t^\alpha$$

$$= \left[(1-\tau_d^*)^{\frac{1}{1-\alpha}}A_d + (1-\tau_c^*)^{\frac{1}{1-\alpha}}A_c\right]^{1-\alpha} k_t^\alpha \left(\frac{1-\alpha}{1+\phi(E_t)}\left[\frac{1}{1+\tau_t} + \phi(E_t)\right]M + \alpha^2\frac{1-\tau_{kt}}{1+\tau_t}\right)$$

By substituting (68) and (69) into the last expression we have

$$c_{yt} + \phi(E_{t-1})c_{ot} + k_{ct+1} + k_{dt+1} = \frac{(1-\tau_d^*)^{\frac{\alpha}{1-\alpha}}A_d + (1-\tau_c^*)^{\frac{\alpha}{1-\alpha}}A_c}{\left[(1-\tau_d^*)^{\frac{1}{1-\alpha}}A_d + (1-\tau_c^*)^{\frac{1}{1-\alpha}}A_c\right]^\alpha} k_t^\alpha = A_c^{1-\alpha}k_{ct}^\alpha + A_c^{1-\alpha}k_{ct}^\alpha$$

since under the taxes (τ_c^*, τ_d^*), $k_{ct} = \frac{(1-\tau_c^*)^{\frac{1}{1-\alpha}}A_c k_t}{(1-\tau_d^*)^{\frac{1}{1-\alpha}}A_d + (1-\tau_c^*)^{\frac{1}{1-\alpha}}A_c}$ and $k_{dt} = \frac{(1-\tau_d^*)^{\frac{1}{1-\alpha}}A_d k_t}{(1-\tau_d^*)^{\frac{1}{1-\alpha}}A_d + (1-\tau_c^*)^{\frac{1}{1-\alpha}}A_c}$; i.e. the feasibility condition is satisfied.

Under the tax policy above, for given k_{cT}, k_{dT}, E_T which are well determined in period T, the dynamics of k_{ct+1}, k_{dt+1}, E_{t+1} from $t+1 \geq T+1$ onwards are

$$k_{ct+1} = \frac{\phi(E_t)}{1+\phi(E_t)}\frac{1+\phi(E^*)}{\phi(E^*)}\frac{\alpha A_c}{\left[m^{\frac{1}{1-\alpha}}A_d + A_c\right]^\alpha}(k_{ct}+k_{dt})^\alpha \tag{74}$$

$$k_{dt+1} = \frac{\phi(E_t)}{1+\phi(E_t)}\frac{1+\phi(E^*)}{\phi(E^*)}\frac{\alpha m^{\frac{1}{1-\alpha}}A_d}{\left[m^{\frac{1}{1-\alpha}}A_d + A_c\right]^\alpha}(k_{ct}+k_{dt})^\alpha \tag{75}$$

$$E_{t+1} = E_t + \psi(E_t) - \xi k_{dt+1} \tag{76}$$

which only depend on m.

We prove next that for all $t \geq T$, it always holds that $E_{t+1} > \hat{E}$. Indeed, the rule of capital allocation in any period $t \geq T$ onwards is

$$\frac{k_{ct}}{A_c} = \frac{k_{dt}}{m^{\frac{1}{1-\alpha}}A_d} \quad \Rightarrow \quad k_{ct} + k_{dt} = \frac{m^{\frac{1}{1-\alpha}}A_d + A_c}{m^{\frac{1}{1-\alpha}}A_d}k_{dt} \tag{77}$$

Hence the dynamics of k_{dt+1}, E_{t+1} can now be characterized by

$$k_{dt+1} = \frac{\phi(E_t)}{1+\phi(E_t)}\frac{1+\phi(E^*)}{\phi(E^*)}\alpha m A_d^{\frac{1}{1-\alpha}}k_{dt}^\alpha$$

$$E_{t+1} = E_t + \psi(E_t) - \xi k_{dt+1}$$

Since $k_{dT} < \frac{\psi(\hat{E})}{\xi}$, then (by assumption $(A4)$)

$$k_{dT+1} = \frac{\phi(E_T)}{1+\phi(E_T)} \frac{1+\phi(E^*)}{\phi(E^*)} \alpha m A_d^{\frac{1}{1-\alpha}} k_{dT}^{\alpha} \leq \frac{\phi(E_T)}{1+\phi(E_T)} \frac{1+\phi(E_+^*)}{\phi(E_+^*)} \alpha m A_d^{\frac{1}{1-\alpha}} k_{dT}^{\alpha}$$

$$< m \left[\frac{\psi(\hat{E})}{\xi}\right]^{1-\alpha} \left[\frac{\psi(\hat{E})}{\xi}\right]^{\alpha} \leq \frac{\psi(\hat{E})}{\xi}$$

since $m \leq 1$, $E^* \geq E_+^*$, as proven in Lemma 6, and $E_T < \bar{E}$.

Hence (note that $\psi'(E) > -1\ \forall E$),

$$E_{T+1} = E_T + \psi(E_T) - \xi k_{dT+1} > \hat{E} + \psi(\hat{E}) - \xi k_{dT+1} \geq \hat{E}$$

So by induction, for all $t \geq T$ we have $E_{t+1} > \hat{E}$.

The budget-balanced competitive steady state under the stationary policy $(\tau_c^*, \tau_d^*, \tau_k, \tau)$ is characterized by

$$\frac{c_o}{c_y} = 1 \tag{78}$$

$$k_c = \left[\alpha \frac{\phi(E)}{1+\phi(E)} \frac{1+\phi(E^*)}{\phi(E^*)}\right]^{\frac{1}{1-\alpha}} A_c \tag{79}$$

$$k_d = \left[\alpha m \frac{\phi(E)}{1+\phi(E)} \frac{1+\phi(E^*)}{\phi(E^*)}\right]^{\frac{1}{1-\alpha}} A_d \tag{80}$$

$$A_c^{1-\alpha} k_c^{\alpha} + A_d^{1-\alpha} k_d^{\alpha} = [1+\phi(E)] c_o + k_c + k_d \tag{81}$$

$$\psi(E) = \xi k_d \tag{82}$$

while the first-best steady state is characterized by

$$\frac{c_o^*}{c_y^*} = 1 \tag{83}$$

$$k_c^* = \alpha^{\frac{1}{1-\alpha}} A_c \tag{84}$$

$$k_d^* = (\alpha m)^{\frac{1}{1-\alpha}} A_d \tag{85}$$

$$A_c^{1-\alpha} k_c^{*\alpha} + A_d^{1-\alpha} k_d^{*\alpha} = [1+\phi(E^*)]\, c_o^* + k_c^* + k_d^* \tag{86}$$

$$\psi(E^*) = \xi k_d^* \tag{87}$$

So, in order for the competitive equilibrium steady state (c_y, c_o, k_c, k_d, E) under the stationary and balanced tax policy $(\tau_c^*, \tau_d^*, \tau_k, \tau)$ to be the first-best steady state, it is necessary and sufficient to prove that $E = E^*$. The two equations (80) and (82) give us

$$\psi(E)=\xi\left[\alpha m\frac{\phi(E)}{1+\phi(E)}\frac{1+\phi(E^*)}{\phi(E^*)}\right]^{\frac{1}{1-\alpha}}A_d=\left[\frac{\phi(E)}{1+\phi(E)}\frac{1+\phi(E^*)}{\phi(E^*)}\right]^{\frac{1}{1-\alpha}}\psi(E^*)$$

since from (85) and (87) we have $\xi(\alpha m)^{\frac{1}{1-\alpha}}A_d=\xi k_d^*=\psi(E^*)$, then,

$$\psi(E)\left[\frac{1+\phi(E)}{\phi(E)}\right]^{\frac{1}{1-\alpha}}=\psi(E^*)\left[\frac{1+\phi(E^*)}{\phi(E^*)}\right]^{\frac{1}{1-\alpha}} \tag{88}$$

Since, for $E>\hat{E}$ the function $\varphi(E)=\psi(E)\left[\frac{1+\phi(E)}{\phi(E)}\right]^{\frac{1}{1-\alpha}}$ is monotonically decreasing in E because

$$\varphi'(E)=\left[\frac{1+\phi(E)}{\phi(E)}\right]^{\frac{1}{1-\alpha}}\left[\psi'(E)-\frac{\psi(E)\phi'(E)}{\phi(E)[1+\phi(E)](1-\alpha)}\right]<0\quad\forall E>\hat{E}$$

That is to say, from (88) we have $E=E^*$.

We complete the proof by showing that under the policy above, the first-best steady state that the economy attains is locally stable. Indeed, under the policy above the Jacobian matrix J_τ^* of the dynamic system $(k_t,E_t)_t$, which can be derived from equations (74) (75) and (76), around the first-best steady state is

$$J_\tau^*=\begin{pmatrix}\alpha & \frac{\phi'(E^*)}{\phi(E^*)[1+\phi(E^*)]}\left(\frac{\alpha A_c}{\left[m^{\frac{1}{1-\alpha}}A_d+A_c\right]^\alpha}\right)^{\frac{1}{1-\alpha}}\\ -\alpha\psi(E^*)\left(\frac{\left[m^{\frac{1}{1-\alpha}}A_d+A_c\right]^\alpha}{\alpha A_c}\right)^{\frac{1}{1-\alpha}} & 1+\psi'(E^*)-\frac{\psi(E^*)\phi'(E^*)}{\phi(E^*)[1+\phi(E^*)]}\end{pmatrix}$$

We have

$$\det(J_\tau^*)=\alpha[1+\psi'(E^*)]\qquad\text{and}\qquad Tr(J_\tau^*)=\alpha+1+\psi'(E^*)-\frac{\psi(E^*)\phi'(E^*)}{\phi(E^*)[1+\phi(E^*)]}$$

which are determined exactly the same as the determinant and trace of the Jacobian matrix J in Subsection 5.2.

Since $E^*\in(\hat{E},\bar{E})$ then the local stability of the first-best steady state is guaranteed as discussed in Subsection 5.2. Q.E.D.

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